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Existence results for nonlinear pseudoparabolic problems

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ABSTRACT

In this paper, we are interested in nonlinear pseudoparabolic problems of type:

$$f\ (t, x, \partial_t u) - \mathrm{Div}[a(x, u, \partial_t u)\nabla u] - \mathrm{Div}[b(x, u, \partial_t u)\nabla \partial u_t] = g.$$

f is a nondecreasing continuous function with respect to its third argument, a is bounded and b is positive and bounded.

The result of existence is proved thanks to a time discretization scheme. Then, we derive some applications to the equation of Barenblatt, a degenerate case and differential inclusions. Finally, some numerical illustrations are proposed.

1. Introduction

In this paper, we are interested in nonlinear pseudoparabolic problems, *i.e.*, when the time derivative of the unknown is present in the main operator. The study of such types of problems is not new; let us cite the works of:

Sobolev in mechanics and Physics [1]. In the linear case, one gets $\partial_t u - a\Delta u - b\Delta\partial_t u = g$ and one talks about the equation of Sobolev.

Barenblatt et al. [2] concerning seepage of homogeneous liquids in fissured rocks, and [3,4] concerning fluid flow where the equations are: $\partial_t u = \partial_x \varphi(\partial_x u) + \tau\partial^2_{xt}\psi(\partial_x u)$ or $\partial_t u + \partial_x[\varphi(u) + \tau\partial_t\varphi(u)] = \partial^2_{xx}[\psi(u) + \tau\partial_t\psi(u)]$.

Showalter [5] (p. 202 & 235) for hysteretic effects in fluid flows, and Showalter et al. [6,7], where the authors have studied the problem: $M\partial_t u + Lu = f$, where M and L are differential operators. Note that if M^{-1} exists, then the problem reads $\partial_t u + M^{-1}Lu = M^{-1}f$.

Considering flows in porous media, Hulshof and King [8] have introduced dynamic effects in the saturation-pressure relation and have proposed the pseudo-parabolic formulation: $\partial_t u = \partial_x(u^\alpha\partial_x u) + \partial_x(u^\beta\partial^2_{xt}u)$. Then, thanks to a notion of dynamic capillarity pressure, Cuesta and Hulshof [9] and Garcia-Azorero and De Pablo [10] were interested in the pseudoparabolic problems: $\partial_t u = \partial^2_{xx}u + 2u\partial_x u + \epsilon^2\partial^3_{xxt}u$ and $\partial_t u = \partial^2_{xx}\varphi(u) + \tau^2\partial^3_{xxt}\varphi(u)$.

Pseudoparabolic operators have also been used in singular perturbation techniques. Let us cite, for example:

Ewing [11,12] for the perturbation of a backward parabolic operator;

Plotnikov [13] where, assuming that φ is not *a priori* nondecreasing, the author has studied the limit when ϵ goes to 0 in $\partial_t u = \Delta\varphi(u) + \epsilon\Delta\partial_t u$ (see [14] p. 426–430 too);

van Duijn et al. [15] where the following singular perturbation $\partial_t u + \partial_x f(u) = \epsilon\partial^2_{xx}u + \epsilon^2\tau\partial^3_{xxt}u$ has been considered. They then noticed that the corresponding solution is not an entropy solution in the sense of Oleinik of the equation of Burgers.

Without trying to be exhaustive, let us cite some other works with pseudoparabolic equations: Ang and Tran [16] and Benjamin et al. [17] concerning long waves in nonlinear dispersive systems; Bouziani and Merazga [18] for dynamics of

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moisture transfer in a subsoil layer; Düll [19] for solvent uptake in polymeric solids; Kaikina [20] concerning global solutions; Korpusov [21] for quasistationary processes in dispersionless conducting media; Padrón [22] in population dynamics and Sviridyuk and Karamova [23] for semiconducting plasma.

As mentioned by Showalter in [5], memory effects in nonlinear pseudoparabolic problems prevent the existence of easy results of uniqueness. On this subject, let us cite for example [24] where a transposition method of Holmgren's type is proposed and Antontsev et al. [25] where a L^p ($p > 2$) regularity is used to prove the uniqueness of the solution of the problem $\partial_t u - \mathrm{div}[\lambda(u)a(\partial_t u + E)\nabla u] - \tau \mathrm{div}[\lambda(u)\nabla\partial_t u] = 0$.

Unlike many of the above mentioned papers, we will be interested in the sequel by equations involving nonlinear functions of $\partial_t u$. Indeed, this paper is part of a sequel of papers of one of the authors, devoted to a model of a sedimentary basin in stratigraphic Geology (see [26–28,25,29–32]). This model is based on a mass conservation equation: $\partial_t u + \mathrm{Div}\{\vec{q}\} = 0$, where the flux $\vec{q}$ satisfies a "Darcy ($\tau = 0$)" or a dynamic "Darcy–Barenblatt ($\tau > 0$)" law: $\vec{q} = -\lambda(\nabla u + \tau\nabla\partial_t u)$.

Moreover, the "weather limited" constraint $\partial_t u + E \geq 0$ is imposed in the domain, so that the model becomes $0 \in \partial_t u - \mathrm{Div}[H(\partial_t u + E)\nabla(u + \tau\partial_t u)]$ where H denotes the graph of Heaviside. Hence our interest for equations of type

$$f(t, x, \partial_t u) - \mathrm{div}[a(x, u, \partial_t u)\nabla u + b(x, u, \partial_t u)\nabla\partial_t u] = g. \tag{1}$$

Let us mention other works in the literature where such kinds of nonlinear terms are used. Let us start with the classical equation of Barenblatt given by $f(\partial_t u) + Au = 0$ where $f(x) = x + \gamma|x|$ and $Au = -\Delta u$ in [33]; $f(x) = x + \gamma x^+$ and $Au = -\Delta u^m$ in [34] or $f(x) = |x|^{m-1}(x + \gamma|x|)$ and $Au = -\Delta_p u$ in [35].

In these papers, the authors were mainly interested in self-similar solutions.

In [36] (p. 81 *sqq.*), M. Ptashnyk has considered problems of reaction–diffusion of biological, chemical or physical substances modeled by:

$$f(t, x, \partial_t u) - \mathrm{div}[a(t, x)h(u)\nabla u] - \mathrm{div}[b(t, x, \nabla\partial_t u)] = g(t, x, u).$$

The author has proved the existence of a solution by using a singular perturbation method by a second order hyperbolic operator and monotonocity arguments. This fails when the function a depends on $\partial_t u$.

A similar technique has been proposed by Colli et al. in [37] for the following differential inclusion: $g \in \mu\partial_t\chi + \alpha(\partial_t\chi) - \delta\Delta\partial_t\chi - \nu\Delta\chi + \beta(\chi)$, and by Schimperna et al. [38] and Segatti [39] for: $g \in A(\partial_t u) + B(u) + h(u)$, where A and B are maximal monotone operators given by the subdifferential of some convex functions.

In order to finish this survey, let us cite Beliaev [40] for the problem: $\partial_t u \in \Delta\{[1 + \kappa\,\mathrm{Sign}(\partial_t u)]u^m\}$, and Schweizer [41] for: $\partial_t u \in \mathrm{Div}[au + b + \gamma\,\mathrm{sign}(\partial_t u)]$, where the authors were interested in a hysteretic porous medium.

At the end of the introduction, one will give the definition of a solution to Problem (1) in $H^1(0, T, H_0^1(\Omega)) = \{u \in L^2(0, T, H_0^1(\Omega)),\ \partial_t u \in L^2(0, T, H_0^1(\Omega))\}$. In a new section, one will prove the main result of this paper, *i.e.* the existence of a solution. Then, one will derive some applications to Barenblatt's equation and to pseudoparabolic differential inclusions. Some numerical illustrations will be given in a last section.

Let us consider in the sequel a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $T > 0$, $Q =]0, T[\times\Omega$ and denote by $\|.\|_{H_0^1(\Omega)} : u \mapsto \|\nabla u\|_{L^2(\Omega)}$ the norm of Poincaré. Then, let us denote by **(H)** the following hypotheses:

$\mathbf{H}_1$: a and b are continuous functions over $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}$ such that:

$$\exists\beta,\ M > 0,\ \forall(x, u, v) \in \overline{\Omega} \times \mathbb{R}^2, \quad |a(x, u, v)| \leq M, \qquad \beta \leq b(x, u, v) \leq M,$$

moreover a (resp. b) is Hölder continuous with exponent θ_a (resp. θ_b) with respect to u, v with $\theta_a,\ \theta_b \geq \frac{1}{2}$.

$\mathbf{H}_2$: f is a Carathéodory function over $Q \times \mathbb{R}$ such that $\tilde{f}(L^2(Q)) \subset L^2(Q)$ where $\tilde{f}$ denotes the operator of Nemytskii[1] associated with f. Moreover, one considers one of the two hypotheses: either $\mathbf{H}_{2,1}$: $u \mapsto f(., u)$ is an increasing function; or $\mathbf{H}_{2,2}$: $u \mapsto f(., u)$ is a nondecreasing function and a, b are Lipschitz continuous.

$\mathbf{H}_3$: $g \in L^2(0, T, H^{-1}(\Omega))$ and $u_0 \in H_0^1(\Omega)$.

Then, one would say that

Definition 1. A solution to Problem (1) is any $u \in H^1(0, T; H_0^1(\Omega))$ such that for any $v \in H_0^1(\Omega)$ and $t \in]0, T[$ a.e.,

$$\int_\Omega \{f(t, x, \partial_t u)v + [a(x, u, \partial_t u)\nabla u + b(x, u, \partial_t u)\nabla\partial_t u]\nabla v\}\mathrm{d}x = \langle g, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \tag{2}$$

with the initial condition $u(0, \cdot) = u_0$.

[1] Cf. Definition 2.

2. Results on Carathéorody functions

Let us recall in this section that (Cf. [42] p. 5 *sqq*, [43], [44] p. 77, [45] p. 407 *sqq*, [46] p. 22 *sqq*).

Definition 2. 1. f is a Carathéodory function on $Q\times\mathbb{R}$ (we say a C-function) if for any real u, $(t,x)\mapsto f(t,x,u)$ is measurable and if for (t,x) a.e. in Q, $u\mapsto f(t,x,u)$ is continuous.
2. If f is a Carathéodory function, for any measurable function $u : Q \to \mathbb{R}$, $f(.,.,u)$ is measurable. One denotes by $\tilde f : u\mapsto \tilde f(u) = f(.,.,u)$. It is the Nemytskii's operator associated with f.

Theorem 1. *If f is a C-function, the following assertions are equivalent:*

$$1.\ \exists a_1, a_2 \in L^2(Q)\times\mathbb{R}^+,\ (t,x)\in Q \text{ a.e},\ \forall u\in\mathbb{R},\ |\tilde f(u)| \le a_1 + a_2|u|, \tag{3}$$

$$2.\ \tilde f(L^2(Q)) \subset L^2(Q),\ \text{i.e. } \forall u\in L^2(Q),\ \tilde f(u)\in L^2(Q),$$

$$3.\ \tilde f \text{ is bounded and continuous in } L^2(Q).$$

Remark 1. Thus, there exists $Z_1 \subset]0,T[$ of full measure such that, for any t in Z_1, $a_1(t,.)\in L^2(\Omega)$, $f(t,.,.)$ is a C-function on $\Omega\times\mathbb{R}$ such that $\tilde f(t,.,.)$ is continuous on $L^2(\Omega)$ and

$$\forall t\in Z_1 \quad x\in\Omega \text{ a.e},\ \forall u\in\mathbb{R},\quad |f(t,x,u)| \le a_1(t,x) + a_2|u|.$$

Let us denote by (ρ_n) the classical mollifier sequence and by

$$f_n(t,x,u) = \int_Q f(s,y,u)\rho_n(t-s,x-y)\mathrm{d}s\mathrm{d}y.$$

Proposition 1. *If f is a C-function with a continuous Nemytskii's operator in $L^2(Q)$, then f_n is continuous on $\overline{Q}\times\mathbb{R}$.*

Proof. This is a consequence of the regularity of the mollifier sequence and the hypothesis on $\tilde f$. □

Remark 2. If f is a C-function satisfying (3) then, $\forall (t,x,u)\in Q\times\mathbb{R}$,

$$|f_n(t,x,u)| \le a_1 * \rho_n(t,x) + a_2|u| \le \|a_1*\rho_n\|_\infty + a_2|u|.$$

Up to a subsequence, still indexed by n, there exists $h\in L^2(Q)$ such that $0\le a_1*\rho_n\le h$ a.e. in Q. Thus, it is possible to assume that (3) holds with the same function a_1, for $\tilde f$ as well as for $\tilde f_n$, in Remark 1.

Lemma 1. *Let f be a C-function satisfying (3). Then, there exists $Z_2\subset Z_1$ ([2]) $\subset]0,T[$ of full measure such that if $(u_n)\subset L^2(\Omega)$ converges weakly to u in $H^1(\Omega)$, then, for any t in Z_2, $f_n(t,.,u_n)$ converges to $f(t,.,u)$ in $L^2(\Omega)$.*

Proof. Let $k\in\mathbb{N}^*$, $\varphi_k\in C(\mathbb{R})$ such that $1_{[-k,k]}\le\varphi_k\le 1_{[-k-1,k+1]}$ and denote by:
$f^k : (t,x,u)\mapsto f(t,x,u)\varphi_k(u)$ and $f_n^k : (t,x,u)\mapsto f_n(t,x,u)\varphi_k(u)$.

Then, thanks to Berliocchi et al. [43] (Remark 5 p. 135), $f^k \in L^2(Q, C[-k-1,k+1])$ and f_n^k converges to f^k in $L^2(Q,C[-k-1,k+1])$. Thus, up to a subsequence still denoted by (f_n^k), there exists $Z_2\subset Z_1 \subset]0,T[$ of full measure such that, for any t in Z_2 and any positive integer k, $f_n^k(t,.,.)$ converges to $f^k(t,.,.)$ in $L^2(\Omega, C[-k-1,k+1])$. Therefore,

$$\int_\Omega |f_n(t,x,u_n(x)) - f(t,x,u_n(x))|^2\,\mathrm{d}x \le C\int_\Omega [\|f_n^k(t,x,.)-f^k(t,x,.)\|_\infty^2 + [a_1^2(t,x)+a_2^2|u_n|^2](1-\varphi_k(u_n))^2]\mathrm{d}x.$$

And,

$$\limsup_{n\to\infty}\int_\Omega |f_n(t,x,u_n(x)) - f(t,x,u_n(x))|^2\,\mathrm{d}x \le C\int_\Omega [a_1^2(t,x)+a_2^2|u|^2](1-\varphi_k(u))^2\,\mathrm{d}x \underset{k\to\infty}{\to} 0.$$

Then, one concludes by using the continuity of the Nemytskii's operator for a fixed t (cf. Remark 1). □

3. Existence of a solution

The method consists, in a first part, in passing to the limit in an implicit time discretization scheme for a continuous function f. In order to do it, one would consider an adapted compactness method. Then, in a second part, one would prove the existence of a solution in the general case.

[2] Cf. Remark 1.

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3.1. The continuous case

Assume that f is a continuous function over $\overline{Q} \times \mathbb{R}$ and that a_1, introduced in (3), is a constant. The idea will be to obtain the general case by regularization, thanks to Proposition 1 and Remark 2.

For any positive integer N, set $h = \frac{T}{N}$ and $t_k = kh$ for $k = 0, 1, \dots, N$. Then, the first result of this section asserts that.

Proposition 2. *Let us assume hypotheses* (H). *Then, for any $h \le \frac{\beta}{M+1}$, there exists a unique sequence (u^k) in $H_0^1(\Omega)$ such that $u^0 = u_0$ and that for any v in $H_0^1(\Omega)$,*

$$\int_\Omega f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) v\,\mathrm{d}x + \int_\Omega a\left(x, u^{k+1}, \frac{u^{k+1}-u^k}{h}\right) \nabla u^{k+1} \nabla v\,\mathrm{d}x + \int_\Omega b\left(x, u^{k+1}, \frac{u^{k+1}-u^{k+1}}{h}\right) \nabla \frac{u^{k+1}-u^k}{h} \nabla v\,\mathrm{d}x = \langle g^{k+1}, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \tag{4}$$

where $g^{k+1}(x) = \int_{kh}^{(k+1)h} g(t,x)\mathrm{d}t$.

Proof. Claim 1. Existence of a solution u^{k+1}.

Assume first that f is a bounded function and denote by Ψ the function defined, for any $S \in H_0^1(\Omega)$, by $\Psi(S) = U$ where U is the unique solution in $H_0^1(\Omega)$ to the variational problem: $U \in H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega f\left(t_{k+1}, x, \frac{S-u^k}{h}\right) v + \left[a\left(x, S, \frac{S-u^k}{h}\right)\nabla U + b\left(x, S, \frac{S-u^{k+1}}{h}\right)\nabla \frac{U-u^k}{h}\right]\nabla v\,\mathrm{d}x = \langle g^{k+1}, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

This problem is well-posed since the hypothesis on h yields $a + b/h \ge 1$, and one is able to prove the existence of a solution u^{k+1} thanks to the fixed point theorem of Schauder–Tikhonov (Cf. [47] Chap. 2 p. 30) applied to Ψ.

In the general case, one has a sequence of solutions (u_n^{k+1}) corresponding with the sequence of bounded functions $f_n = \min[n, \max(-n, f)]$. Then, one passes to the limit since $(u_n^{k+1})_n$ is bounded in $H_0^1(\Omega)$, since $|f_n| \le |f|$ and thanks to Lemma 1.

Claim 2. uniqueness of the solution u^{k+1}.

Let us consider two solutions u^{k+1} and $\hat{u}^{k+1}$, set $w = u^{k+1} - \hat{u}^{k+1}$ and $v = p(w)$ where p is a nondecreasing Lipschitz-continuous function to be made precise later in the proof, with $p(0) = 0$. Then,

$$\begin{aligned}
0 = &\int_\Omega \left[f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) - f\left(t_k, x, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p(w)\mathrm{d}x \\
&+ \int_\Omega \left[a\left(x, u^{k+1}, \frac{u^{k+1}-u^k}{h}\right) + \frac{1}{h} b\left(x, u^{k+1}, \frac{u^{k+1}-u^k}{h}\right)\right] p'(w)|\nabla w|^2 \mathrm{d}x \\
&+ \int_\Omega \left[a\left(x, u^{k+1}, \frac{u^{k+1}-u^k}{h}\right) - a\left(x, \hat{u}^{k+1}, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p'(w)\nabla \hat{u}^{k+1} \nabla w\,\mathrm{d}x \\
&+ \int_\Omega \left[b\left(x, u^{k+1}, \frac{u^{k+1}-u^k}{h}\right) - b\left(x, \hat{u}^{k+1}, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p'(w)\nabla \frac{\hat{u}^{k+1}-u^k}{h} \nabla w\,\mathrm{d}x.
\end{aligned}$$

Then, $h \le \frac{\beta}{M+1}$ and H_1 yield the existence of a positive constant C such that

$$\begin{aligned}
&\int_\Omega \left\{\left[f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) - f\left(t_{k+1}, x, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p(w) + p'(w)|\nabla w|^2\right\}\mathrm{d}x \\
&\le C\int_\Omega |w|^{\theta_a} |\nabla \widehat{u}^{k+1}||\nabla w| p'(w)\mathrm{d}x + C\int_\Omega |w|^{\theta_b}\left|\nabla \frac{\hat{u}^{k+1}-u_0}{h}\right| |\nabla w| p'(w)\mathrm{d}x.
\end{aligned}$$

Since $p' \ge 0$, for another positive constant C, one gets that

$$\begin{aligned}
&\int_\Omega \left\{\left[f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) - f\left(t_{k+1}, x, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p(w) + p'(w)|\nabla w|^2\right\}\mathrm{d}x \\
&\le C\int_\Omega p'(w)\left[|w|^{2\theta_a}|\nabla \widehat{u}^{k+1}|^2 + |w|^{2\theta_b}\left|\nabla \frac{\hat{u}^{k+1}-u_0}{h}\right|^2\right]\mathrm{d}x.
\end{aligned}$$

If, on one hand $H_{2,1}$ is assumed, one sets $p(r) = \ln[\frac{e}{\mu}\min[\mu, \max(r, \frac{\mu}{e})]]$, $\mu > 0$.

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Thus, $p'(w) = \frac{1}{w}\mathbf{1}_{\Omega_\mu}(w)$ where $\Omega_\mu = \{\frac{\mu}{e} < w < \mu\}$, and

$$\int_\Omega \left[f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) - f\left(t_{k+1}, x, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right] p(w)\mathrm{d}x$$
$$\le C\int_{\Omega_\mu}\left[|w|^{2\theta_a-1}|\nabla\widehat{u}^{k+1}|^2 + |w|^{2\theta_b-1}\left|\nabla\frac{\hat{u}^{k+1}-u_0}{h}\right|^2\right]\mathrm{d}x.$$

Passing to the limit with μ to 0^+ leads to

$$\int_\Omega \left[f\left(t_{k+1}, x, \frac{u^{k+1}-u^k}{h}\right) - f\left(t_{k+1}, x, \frac{\hat{u}^{k+1}-u^k}{h}\right)\right]^+ \mathrm{d}x \le 0.$$

Since a similar inequality holds for the negative part, one proves that the solution is unique when $f(t_k, x, .)$ is an increasing function.

If, on the other hand $H_{2,2}$ is assumed, one sets $p(r) = \frac{(r-\mu)^+}{r}$, $\mu > 0$.
Then, $p'(w) = \frac{\mu}{w^2}\mathbf{1}_{\Omega_\mu}(w)$ where $\Omega_\mu = \{\mu < w\}$ and

$$\int_{\Omega_\mu}\frac{|\nabla w|^2}{|w|^2} \le C\int_\Omega\left[|\nabla\widehat{u}^{k+1}|^2 + \left|\nabla\frac{\hat{u}^{k+1}-u_0}{h}\right|^2\right]\mathrm{d}x < \infty.$$

Now, denote by $F(r) = \ln(1 + \frac{(r-\mu)^+}{\mu})$ and remark that the inequality of Poincaré yields

$$\forall\mu > 0, \quad \int_\Omega |F_\mu(u^{k+1} - \widehat{u}^{k+1})|^2\mathrm{d}x \le C.$$

Passing to the limit with μ to 0^+ leads to $u^{k+1} \le \hat{u}^{k+1}$ and to the uniqueness as above. □

In order to prove the existence of a solution, let us introduce some notations: for any sequence (v^k), let us consider, in the sequel of this section, two sequences of functions denoted by v^h and $\widetilde{v}^h$ such that

$$v^h = \sum_{k=0}^{N-1} u^{k+1}\mathbf{1}_{[t_k,t_{k+1}[} \quad \text{and} \quad \widetilde{v}^h = \sum_{k=0}^{N-1}\left[\frac{u^{k+1}-u^k}{h}(t-t_k) + u^k\right]\mathbf{1}_{[t_k,t_{k+1}[}.$$

In particular, $f^h(t, x, u) = \sum_{k=0}^{N-1} f(t_{k+1}, x, u)\mathbf{1}_{[t_k,t_{k+1}[}$ converges to f uniformly in $\overline{Q}\times K$ for any compact $K \subset \mathbb{R}$ and, for t a.e. in $]0, T[$, $f^h(t, ., .)$ converges to $f(t, ., .)$ in $C(\overline{\Omega}\times K)$. Moreover, $g^h = \sum_{k=0}^{N-1} g^{k+1}\mathbf{1}_{[t_k,t_{k+1}[}$ converges to g in $L^2(0, T, H^{-1}(\Omega))$ and, for t a.e. in $]0, T[$, one has that $g^h(t)$ converges to $g(t)$ in $H^{-1}(\Omega)$. Thus, the following *a priori* estimates hold:

Lemma 2. *If* $h < \min[\frac{\beta^2}{8M^2T}, \frac{\beta}{M+1}]$, *then*

1. (u^h) *is bounded in* $L^\infty(0, T; H_0^1(\Omega))$ *and* $(\widetilde{u}^h)$ *in* $H^1(0, T; H_0^1(\Omega))$.
2. $\exists C > 0, \forall t \in [0, T[, \|\widetilde{u}^h(t) - u^h(t)\|_{H_0^1(\Omega)} \le C\sqrt{h}$.
3. t *a.e. in* $]0, T[, \exists C(t) > 0, \|\partial_t\widetilde{u}^h(t)\|_{H_0^1(\Omega)} \le C(t)$.

Proof. Let us consider the test-function $v = \frac{u^{k+1}-u^k}{h}$ in (4). Then, hypotheses (H) lead to (C_P denoting the constant of Poincaré):

$$\beta\left\|\frac{u^{k+1}-u^k}{h}\right\|^2_{H_0^1(\Omega)} \le M\|u^{k+1}\|_{H_0^1(\Omega)}\left\|\frac{u^{k+1}-u^k}{h}\right\|_{H_0^1(\Omega)}$$
$$+\left[\sqrt{1+C_P^2}\|g^{k+1}\|_{H^{-1}(\Omega)} + C_P a_1\sqrt{meas(\Omega)}\right]\left\|\frac{u^{k+1}-u^k}{h}\right\|_{H_0^1(\Omega)}.$$

Writing, $\nabla u^{k+1} = \nabla u_0 + \sum_{p=0}^k(h\nabla\frac{u^{p+1}-u^p}{h})$, one gets that

$$\beta^2\left\|\frac{u^{k+1}-u^k}{h}\right\|^2_{H_0^1(\Omega)} \le 4M^2h^2\left(\sum_{p=0}^k\left\|\frac{u^{p+1}-u^p}{h}\right\|_{H_0^1(\Omega)}\right)^2 + 4M^2\|u_0\|^2_{H_0^1(\Omega)}$$
$$+4(1+C_P^2)\|g^{k+1}\|^2_{H^{-1}(\Omega)} + 4C_P^2a_1^2 meas(\Omega)$$
$$\le 4M^2(k+1)h^2\sum_{p=0}^k\left\|\frac{u^{p+1}-u^p}{h}\right\|^2_{H_0^1(\Omega)} + 4M^2\|u_0\|^2_{H_0^1(\Omega)}$$
$$+4(1+C_P^2)\|g^{k+1}\|^2_{H^{-1}(\Omega)} + 4C_P^2a_1^2 meas(\Omega).$$

Then, since $k \le N-1$ and $h < \frac{\beta^2}{8M^2T}$, one has

$$\frac{\beta^2}{h}\|u^{k+1}-u^k\|^2_{H_0^1(\Omega)} \le 8M^2Th\sum_{p=0}^{k-1}\frac{1}{h}\|u^{p+1}-u^p\|^2_{H_0^1(\Omega)} + 8M^2h\|u_0\|^2_{H_0^1(\Omega)} + 8(1+C_P^2)h\|g^{k+1}\|^2_{H^{-1}(\Omega)} + 8hC_P^2a_1^2 meas(\Omega). \quad (5)$$

By summing over k, this leads us to

$$\sum_{k=0}^{N-1}\frac{1}{h}\|u^{k+1}-u^k\|^2_{H_0^1(\Omega)} \le \frac{8M^2T}{\beta^2}\left[\sum_{k=0}^{N-1}\sum_{p=0}^{k-1}\|u^{p+1}-u^p\|^2_{H_0^1(\Omega)} + \|u_0\|^2_{H_0^1(\Omega)}\right] + \frac{8(1+C_P^2)}{\beta^2}\|g^h\|^2_{L^2(0,T,H^{-1}(\Omega))} + \frac{8TC_P^2a_1^2 meas(\Omega)}{\beta^2}.$$

Since (g^h) is bounded in $L^2(0,T,H^{-1}(\Omega))$, there exists a positive constant $\widetilde{C}$ such that

$$\sum_{k=0}^{N-1}\frac{1}{h}\|u^{k+1}-u^k\|^2_{H_0^1(\Omega)} \le \frac{8M^2Th}{\beta^2}\sum_{k=0}^{N-1}\sum_{p=0}^{k-1}\frac{1}{h}\|u^{p+1}-u^p\|^2_{H_0^1(\Omega)} + \widetilde{C}$$

and the discrete Gronwall's lemma (cf. [48] p. 165) yields

$$\|\partial_t\widetilde{u}^h\|^2_{L^2(0,T,H_0^1(\Omega))} = \sum_{k=0}^{N-1}\frac{1}{h}\|u^{k+1}-u^k\|^2_{H_0^1(\Omega)} \le \widetilde{C}\exp\left(\frac{8M^2T^2}{\beta^2}\right) := C_1.$$

Moreover,

$$\begin{aligned}\|u^{k+1}\|^2_{H_0^1(\Omega)} &\le 2h^2\left(\sum_{p=0}^{k}\left\|\frac{u^{p+1}-u^p}{h}\right\|_{H_0^1(\Omega)}\right)^2 + 2\|u_0\|^2_{H_0^1(\Omega)}\\ &\le 2(k+1)h\sum_{p=0}^{k}\frac{1}{h}\left\|u^{p+1}-u^p\right\|^2_{H_0^1(\Omega)} + 2\|u_0\|^2_{H_0^1(\Omega)}\\ &\le 2TC_1 + 2\|u_0\|^2_{H_0^1(\Omega)} := C_2^2.\end{aligned}$$

Thus, thanks to the definitions of u^h and $\widetilde{u}^h$, one has $\|u^h\|_{L^\infty(0,T,H_0^1(\Omega))} \le C_2$, $\|\widetilde{u}^h\|_{L^\infty(0,T,H_0^1(\Omega))} \le 2C_2 := C_3$ and $\|\widetilde{u}^h(t)-u^h(t)\|_{H_0^1(\Omega)} \le \sqrt{h}C_1$ for any t in $[0,T]$.

Then, using again (5), one proves that there exists $C_4 > 0$ such that, for any $t \in]t_k, t_{k+1}[$,

$$\|\partial_t\widetilde{u}^h(t)\|^2_{H_0^1(\Omega)} \le C_4[1 + \|g^h(t)\|^2_{H^{-1}(\Omega)} + a_1^2 meas(Q)],$$

and the lemma holds since for $t \in]0,T[$ a.e., $g^h(t)$ converges to $g(t)$ in $H^{-1}(\Omega)$. □

Let us prove now the main result of this section.

Theorem 2. *There exists a solution to Problem (1) in the sense of Definition 1.*

Proof. Using the above notations, for t a.e. in $]0,T[$ and any v in $H_0^1(\Omega)$, one has that

$$\int_\Omega [f^h(t,x,\partial_t\widetilde{u}^h)v + a(x,u^h,\partial_t\widetilde{u}^h)\nabla u^h\nabla v + b(x,u^h,\partial_t\widetilde{u}^h)\nabla\partial_t\widetilde{u}^h\nabla v]dx = \langle g^h, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Then, thanks to Lemma 2, there exists u in $H^1(0,T;H_0^1(\Omega))$ and a subsequence denoted in the way, such that

$$\widetilde{u}^h \rightharpoonup u \quad \text{in } H^1(0,T;H_0^1(\Omega)) \text{ and } \forall t\in[0,T], \qquad \widetilde{u}^h(t), u^h(t) \rightharpoonup u(t) \text{ in } H_0^1(\Omega).$$

Moreover, there exists a measurable set $Z \subset]0,T[$ with $\mathcal{L}(]0,T[\backslash Z) = 0$ and, for any $t \in Z$, there exists $C = C(t) \ge 0$, such that $\|\partial_t\widetilde{u}^h(t)\|_{H_0^1(\Omega)} \le C$.

Hence, there exist $\xi(t) \in H_0^1(\Omega)$, $w \in L^2(\Omega)$ and a subsequence denoted by $(\partial_t\widetilde{u}^{h_t}(t))$ such that $\partial_t\widetilde{u}^{h_t}(t) \rightharpoonup \xi(t)$ in $H_0^1(\Omega)$, $\partial_t\widetilde{u}^{h_t}(t) \to \xi(t)$ in $L^2(\Omega)$ and a.e. in Ω and $|\partial_t\widetilde{u}^{h_t}(t)| \le w$ a.e. in Ω.

Thanks to H_2, and to Lebesgue's theorem, for any $v \in H_0^1(\Omega)$,

$$a(x,u^{h_t}(t),\partial_t\widetilde{u}^{h_t}(t))\nabla v \to a(x,u(t),\xi(t))\nabla v \quad \text{in } L^2(\Omega)^d,$$
$$b(x,u^{h_t}(t),\partial_t\widetilde{u}^{h_t}(t))\nabla v \to b(x,u(t),\xi(t))\nabla v \quad \text{in } L^2(\Omega)^d.$$

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Moreover, $f^{h_t}(t,x,\partial_t\widetilde{u}^{h_t}(t))$ converges to $f(t,x,\xi(t))$ a.e. in Ω and $|f^{h_t}(t,x,\partial_t\widetilde{u}^{h_t}(t))| \leq a_1 + a_2 w$. Thus, one gets that

$$f^{h_t}(t,x,\partial_t\widetilde{u}^{h_t}(t)) \to f(t,x,\xi(t)) \quad \text{in } L^2(\Omega). \tag{6}$$

Thus, passing to the limit, $\xi(t)$ is a solution in $H_0^1(\Omega)$ to the variational problem: $U \in H_0^1(\Omega)$ such that $\forall v$ in $H_0^1(\Omega)$,

$$\int_\Omega [f(t,x,U)v + a(x,u(t),U)\nabla u(t)\nabla v + b(x,u(t),U)\nabla U\nabla v]\mathrm{d}x = \langle g(t), v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}. \tag{7}$$

But, such a solution is unique. Indeed, one has just to adapt the method detailed in the proof of Proposition 2. Therefore, $\partial_t\widetilde{u}^h(t)$ converges weakly in $H_0^1(\Omega)$ to $\xi(t)$.

Let us denote by $\xi :]0,T[\to H_0^1(\Omega), t \mapsto \xi(t)$. It is then a weakly-measurable function, thus a measurable function since $H_0^1(\Omega)$ is separable (cf. Pettis's theorem [49]).

Moreover, for any v in $H_0^1(\Omega)$ and any t a.e. in $]0,T[$,

$$\int_\Omega \nabla\partial_t\widetilde{u}^h(t).\nabla v\,\mathrm{d}x \to \int_\Omega \nabla\xi(t).\nabla v\,\mathrm{d}x \quad\text{and}\quad \left|\int_\Omega \nabla\partial_t\widetilde{u}^h(t).\nabla v\,\mathrm{d}x\right| \leq \|\partial_t\widetilde{u}^h(t)\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}.$$

Then, thanks to Lemma 1.3 p. 12 of [50], for any α in $L^2(0,T)$, $\int_0^T\int_\Omega \alpha(t)\nabla\partial_t\widetilde{u}^h(t).\nabla v\,\mathrm{d}x\mathrm{d}t$ converges to $\int_0^T\int_\Omega \alpha(t)\nabla\xi(t).\nabla v\,\mathrm{d}x\mathrm{d}t$.

Since $(\partial_t\widetilde{u}^h)$ is a bounded sequence in $L^2(0,T;H_0^1(\Omega))$, a density argument leads to the weak convergence in $L^2(0,T;H_0^1(\Omega))$ of $\partial_t\widetilde{u}^h$ to ξ.

Thus, $\partial_t u = \xi$ and there exists a solution. □

3.2. The general case

One proposes now to establish the result of existence of Theorem 2 when the Carathéodory function f satisfies (H). In order to prove it, one denotes by f_n the function $f_n(t,x,u) = \int_Q f(s,y,u)\rho_n(t-s,x-y)\mathrm{d}s\mathrm{d}y$ where (ρ_n) is the usual mollifier sequence.

Thanks to the previous section, there exists a sequence of solutions (u_n) when f is replaced by f_n, *i.e.*.

Lemma 3. *For any positive integer n, there exists u_n in $H^1(0,T;H_0^1(\Omega))$ such that for $t \in]0,T[$ a.e. and any v in $H_0^1(\Omega)$,*

$$\int_\Omega [f_n(t,x,\partial_t u_n)v + a(x,u_n,\partial_t u_n)\nabla u_n\nabla v + b(x,u_n,\partial_t u_n)\nabla\partial_t u_n\nabla v]\mathrm{d}x = \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$$

with $u_n(0,\cdot) = u_0$.

Moreover, the following estimates hold:

1. *(u_n) is bounded in $H^1(0,T;H_0^1(\Omega))$.*
2. *t a.e. in $]0,T[$, $\exists C(t) > 0$, $\|\partial_t u_n(t)\|_{H_0^1(\Omega)} \leq C(t)$.*

Proof. Let us test the above equation with $\partial_t u_n$. Thus,

$$\int_\Omega [f_n(t,x,\partial_t u_n) - f_n(t,x,0)]\partial_t u_n + a(x,u_n,\partial_t u_n)\nabla u_n\nabla\partial_t u_n\mathrm{d}x + \int_\Omega b(x,u_n,\partial_t u_n)|\nabla\partial_t u_n|^2\mathrm{d}x$$
$$= \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega f_n(t,x,0)\partial_t u_n\mathrm{d}x.$$

Then,

$$\int_\Omega \left[\frac{\beta}{2}|\nabla\partial_t u_n|^2 - \frac{M^2}{2\beta}|\nabla u_n|^2\right]\mathrm{d}x \leq \langle g, \partial_t u_n\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega f_n(t,x,0)\partial_t u_n\mathrm{d}x$$

and

$$\int_\Omega |\nabla\partial_t u_n|^2\mathrm{d}x \leq \frac{2M^2}{\beta^2}\int_\Omega\left[t\int_0^t |\nabla\partial_t u_n|^2\mathrm{d}s + |\nabla u_0|^2\right]\mathrm{d}x$$
$$+ \frac{2\|g\|_{H^{-1}(\Omega)}}{\beta}\|\partial_t u_n\|_{H^1(\Omega)} + \frac{2\|f_n(t,x,0)\|_{L^2(\Omega)}}{\beta}\|\partial_t u_n\|_{L^2(\Omega)}.$$

Then, thanks to Remark 2 and Young's inequality, if one denotes $k = \frac{4TM^2}{\beta^2}$,

$$\int_0^t \|\nabla\partial_t u_n\|^2_{L^2(\Omega)}\mathrm{d}s \leq k\int_0^t\int_0^s \|\nabla\partial_t u_n\|^2_{L^2(\Omega)}\mathrm{d}\tau\mathrm{d}s + k\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{4\|g\|^2_{L^2(0,T,H^{-1}(\Omega))}}{\beta} + \frac{4\|a_1\|^2_{L^2(Q)}}{\beta}.$$

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Then, Gronwall's lemma yields, for $t \in]0, T[$ a.e.,

$$\int_0^t \|\nabla \partial_t u_n\|^2_{L^2(\Omega)} \mathrm{d}s \leq C_1 = \left[k\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{4\|g\|^2_{L^2(0,T,H^{-1}(\Omega))}}{\beta} + \frac{4\|a_1\|^2_{L^2(Q)}}{\beta}\right] e^{kT},$$

$$\|\nabla \partial_t u_n\|^2_{L^2(\Omega)} \leq kC_1 + \frac{4M^2}{\beta^2}\|\nabla u_0\|^2_{L^2(\Omega)} + \frac{4\|g(t)\|^2_{H^{-1}(\Omega)}}{\beta} + \frac{4\|a_1(t)\|^2_{L^2(\Omega)}}{\beta}.$$

And, one concludes by writing $u_n(t) = u_0 + \int_0^t \partial_t u_n(s) \mathrm{d}s$. □

Proof of Theorem 2. One is then able to carry on with the proof of Theorem 2, as presented in the previous section. The only difference concerns the assertion (6), but this last point is studied in Lemma 1. □

4. Applications

One proposes in this section to derive some applications. The first one concerns the equation of Barenblatt, *i.e.* $f(t, x, \partial_t u) - \Delta u = g$.

Then, one proposes to study the case of a degenerate version of the pseudoparabolic problem, *i.e.* when b may vanish at some point.

Finally, two applications are proposed. If $f \subset \mathbb{R}^2$ is a graph of domain $\mathbb{R}$ in the first one, and for the penalization of a constraint in the second one.

4.0.1. Application to Barenblatt's equation

Barenblatt's equation, in a simplified form, is $f(\partial_t u) - \Delta u = g$, where $f(r) = r$ if $r > 0$ and $f(r) = \alpha r$ $(\alpha > 0)$ if $r \leq 0$, for the initial condition u_0. One would consider in this section the problem, in $H^1_0(\Omega)$:

$$f(t, x, \partial_t u) - \Delta u = g, \qquad u(t = 0) = u_0, \tag{8}$$

where $g \in L^2(Q)$, $u_0 \in H^1_0(\Omega)$ and f is a function that satisfies the hypotheses (H_2), such that the associated Nemystkii's operator $\tilde{f}$ is monotone in $L^2(Q)$. Moreover, one assumes that there exists $\alpha > 0$ such that $\alpha|u|^2 \leq f(t, x, u)u$ for any real u and (t, x) a.e. in Q.

Then, following Lions [50] Th. 2.1 p. 171, since $A = I + \tilde{f}$ is monotone, bounded and continuous in $L^2(Q)$, with $\lim_{\|u\|_{L^2(Q)} \to \infty} \frac{\int_Q A(u)u \mathrm{d}x\mathrm{d}t}{\|u\|_{L^2(Q)}} = +\infty$, one gets that A is surjective on $L^2(Q)$ and that $\tilde{f}$ is maximal monotone.

The aim of this section is then to prove that.

Theorem 3. *There exists a solution $u \in H^1(Q) \cap L^2(0, T, H^1_0(\Omega))$ to Problem (8) in the sense: $u(0, .) = u_0$ and $\forall v \in H^1_0(\Omega)$, $t \in]0, T[$ a.e.*

$$\int_\Omega f(t, x, \partial_t u)v + \nabla u \nabla v \, \mathrm{d}x = \int_\Omega gv \, \mathrm{d}x.$$

In order to prove the existence of a solution, let us consider a singular perturbation method by a pseudoparabolic operator. Thus, for any positive ϵ, one considers the family of solutions u_ϵ of Problem (1) where $a(x, u, \partial_t u) = 1$ and $b(x, u, \partial_t u) = \epsilon$.

Then, by using the test function $v = \partial_t u_\epsilon$, one gets, for any t,

$$\alpha \int_{]0,t[\times\Omega} |\partial_t u_\epsilon|^2 \, \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u_\epsilon(t)|^2 \, \mathrm{d}x + \epsilon \int_{]0,t[\times\Omega} |\nabla \partial_t u_\epsilon|^2 \, \mathrm{d}x \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2 \, \mathrm{d}x + \int_{]0,t[\times\Omega} g \partial_t u_\epsilon \, \mathrm{d}x\mathrm{d}t.$$

Thus, the sequence (u_ϵ) is bounded in $H^1(Q) \cap L^\infty(0, T; H^1_0(\Omega))$, $(\tilde{f}(\partial_t u_\epsilon))$ and $(\partial_t u_\epsilon)$ are bounded sequences in $L^2(Q)$ and $(\sqrt{\epsilon}\partial_t u_\epsilon)$ is bounded in $L^2(0, T; H^1_0(\Omega))$.

Denote by u a weak limit in $H^1(Q)$, and weak-* in $L^\infty(0, T; H^1_0(\Omega))$, associated with a subsequence still denoted by (u_ϵ) and χ a weak limit in $L^2(Q)$ of $(\tilde{f}(\partial_t u_\epsilon))$, for a same sub-sequence.

On one hand, passing to the limit with ϵ leads to:

$$\chi - \Delta u = g \quad \text{or} \quad \partial_t u - \Delta u = g + \partial_t u - \chi.$$

Since $g + \partial_t u - \chi \in L^2(Q)$ and $u_0 \in H^1_0(\Omega)$, the classical energy equality states

$$\int_Q \chi \partial_t u \, \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u(T)|^2 \, \mathrm{d}x = \frac{1}{2}\int_\Omega |\nabla u_0|^2 \, \mathrm{d}x + \int_Q g\partial_t u \, \mathrm{d}x\mathrm{d}t.$$

On the other hand,

$$\int_Q f(t,x,\partial_t u_\epsilon)\partial_t u_\epsilon \, \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u_\epsilon(T)|^2 - |\nabla u_0|^2 \, \mathrm{d}x \leq \int_Q g\partial_t u_\epsilon \, \mathrm{d}x\mathrm{d}t \, .$$

Moreover, $u_\epsilon(T)$ converges to $u(T)$ in $L^2(\Omega)$ and $(u_\epsilon(T))$ is bounded in $H_0^1(\Omega)$. Thus, it converges weakly in $H_0^1(\Omega)$ and passing to the upper limit yields

$$\limsup_{\epsilon\to 0} \int_Q f(t,x,\partial_t u_\epsilon)\partial_t u_\epsilon \, \mathrm{d}x\mathrm{d}t + \frac{1}{2}\int_\Omega |\nabla u(T)|^2 \, \mathrm{d}x \leq \frac{1}{2}\int_\Omega |\nabla u_0|^2 \, \mathrm{d}x + \int_Q g\partial_t u \, \mathrm{d}x\mathrm{d}t.$$

Therefore, $\limsup_{\epsilon\to 0} \int_Q f(t,x,\partial_t u_\epsilon)\partial_t u_\epsilon \, \mathrm{d}x\mathrm{d}t \leq \int_Q \chi\,\partial_t u \, \mathrm{d}x\mathrm{d}t$ and Brézis [51] Prop. 2.5 p. 27 characterizes the limit $\chi = \tilde{f}(\partial_t u)$ and a solution exists.

4.0.2. On a degenerate version of the pseudoparabolic equation

Let us still assume hypothesis (H), except in H_1 where $\beta \geq 0$ is considered. Thus, the problem degenerates in the free set where $b = 0$. On the other hand, as usual for degenerate problems, other information have to be given concerning the link between a and b. So, one assumes that there exist two positive constants C_1, C_2 and a measurable function c such that

$$\forall (x,u,v) \in \overline{\Omega}\times\mathbb{R}^2, \qquad C_1|a(x,u,v)| \leq c(v) \leq C_2 b(x,u,v).$$

Then, for technical reasons, one assumes that $f(t,x,.)$ is an increasing function and that

$$\begin{aligned}
&\forall (x,u,v) \in \mathbb{R}^{d+2}, \quad b(x,u,v) = \alpha(x,u)\beta(v),\\
&|a[x,u,B^{-1}(v_1)] - a[x,u,B^{-1}(v_2)]| \leq \theta(|v_1 - v_2|),\\
&\quad \text{where } B(v) = \int_0^v \beta(s)\mathrm{d}s \quad \text{and} \quad \int_{0^+}^{\cdot} \frac{\mathrm{d}s}{\theta^2(s)} = +\infty,
\end{aligned}$$

for an increasing modulus of continuity θ such that $\theta(0) = 0$ (one talks about Osgood's property). Finally, one sets $g = f(t,x,0)$. Note that, in particular, there exists $\epsilon_0 > 0$ such that $\alpha \geq \epsilon_0$.

Remark 3. This study is a generalization of the framework proposed by Antontsev et al. in [52] concerning the evolution of sedimentary basins. In that case, the authors set $a(x,u,v) = b(x,u,v) = c(v - E)$ with $c = 0$ on $\mathbb{R}^-$ and $f = Id$. In their paper, the problem has to degenerate in order to satisfy implicitly the constraint $\partial_t u \geq E$ where E is a nonnegative constant.

Our aim in this section is then to prove that.

Theorem 4. *With the above hypothesis, there exists a solution of Problem (1) in the sense of Definition 1.*

Proof. Let us apply the idea of the artificial viscosity method; *i.e.*, for any positive ϵ, one considers the family of solutions (u_ϵ) of Problem (1) where $b_\epsilon = b + \epsilon$ replaces b. Such a family exists since $b_\epsilon \geq \epsilon > 0$ and let us get some *a priori* estimates in order to pass to the limits with ϵ to 0^+.

Since $v = \int_0^{\partial_t u_\epsilon} \frac{\mathrm{d}s}{c(s)+C_2\epsilon}$ is a test-function for (2) (with b_ϵ), one gets that

$$0 = \int_\Omega [f(t,x,\partial_t u_\epsilon) - f(t,x,0)] \int_0^{\partial_t u_\epsilon} \frac{\mathrm{d}s}{c(s)+C_2\epsilon}\mathrm{d}x + \int_\Omega \left[\frac{a(.,u_\epsilon,\partial_t u_\epsilon)}{c(\partial_t u_\epsilon)+C_2\epsilon}\nabla u_\epsilon + \frac{b_\epsilon(.,u_\epsilon,\partial_t u_\epsilon)}{c(\partial_t u_\epsilon)+C_2\epsilon}\nabla\partial_t u_\epsilon\right]\nabla\partial_t u_\epsilon \mathrm{d}x.$$

Therefore,

$$\frac{C_1}{C_2}\|\nabla\partial_t u_\epsilon\|_{L^2(\Omega)} \leq \|\nabla u_\epsilon\|_{L^2(\Omega)} \leq \left\|\int_0^t \nabla\partial_t u_\epsilon \mathrm{d}s\right\|_{L^2(\Omega)} + \|\nabla u_0\|_{L^2(\Omega)},$$

and

$$\|\nabla\partial_t u_\epsilon\|^2_{L^2(\Omega)} \leq \frac{2C_2^2 T}{C_1^2}\int_0^t \|\nabla\partial_t u_\epsilon\|^2_{L^2(\Omega)}\mathrm{d}s + \frac{2C_2^2}{C_1^2}\|\nabla u_0\|^2_{L^2(\Omega)}.$$

Then, the sequence (u_ϵ) is bounded in $W^{1,\infty}(0,T;H_0^1(\Omega))$ and the proof detailed in the main section yields the existence of a solution, as soon as one will have explained the result of uniqueness, in the degenerate case, of the solution $\xi(t)$ in $H_0^1(\Omega)$ to the variational problem: $U \in H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$,

$$\int_\Omega \{f(t,x,U)v + a(x,u(t),U)\nabla u(t)\nabla v + \alpha(x,u(t))\nabla B(U)\nabla v\}\mathrm{d}x = \int_\Omega g v \mathrm{d}x.$$

Let us consider $\xi(t)$ and $\hat{\xi}(t)$ two solutions and set $v = p_\mu[B(\xi(t)) - B(\widehat{\xi}(t))]$, where, for any $\mu > 0$, one denotes by $\epsilon(\mu)$ the positive number such that $\int_{\epsilon(\mu)}^{\mu} \frac{dt}{\theta^2(t)} = 1$ and $p_\mu(r) = \int_{\max[\min(r,\mu),\epsilon(\mu)]}^{\mu} \frac{dt}{\theta^2(t)}$. Then, one has that

$$\int_\Omega [f(t,x,\xi(t)) - f(t,x,\widehat{\xi}(t))] p_\mu[B(\xi(t)) - B(\widehat{\xi}(t))]\mathrm{d}x + \int_\Omega p'_\mu(B(\xi(t)) - B(\widehat{\xi}(t)))\alpha(x,u(t))|\nabla(B(\xi(t)) - B(\widehat{\xi}(t)))|^2\mathrm{d}x \le \int_\Omega p'_\mu[B(\xi(t)) - B(\widehat{\xi}(t))]\theta[B(\xi(t)) - B(\widehat{\xi}(t))]|\nabla u(t)||\nabla[B(\xi(t)) - B(\widehat{\xi}(t))]|\mathrm{d}x$$

and

$$\int_\Omega [f(t,x,\xi(t)) - f(t,x,\widehat{\xi}(t))] p_\mu[B(\xi(t)) - B(\widehat{\xi}(t))]\mathrm{d}x \le C(\epsilon_0)\int_{\{\epsilon(\mu) < B(\xi(t)) - B(\widehat{\xi}(t)) < \mu\}} |\nabla u(t)|^2\mathrm{d}x.$$

By passing to the limit with μ to 0^+, one concludes that $\xi(t) \le \widehat{\xi}(t)$. Since the same proof asserts $\xi(t) \ge \widehat{\xi}(t)$ too, the solution is unique and the same method holds. □

4.0.3. *The case of a graph*

Let us assume in this section that f depends on the only variable $\partial_t u$ and that $f \subset \mathbb{R}^2$ is a maximal monotone graph such that $f[L^2(Q)] \subseteq L^2(Q)$ in a sense to be clarified. Then, Dom f is $\mathbb{R}$ and there exists a nondecreasing map $\bar{f}$ on $\mathbb{R}$ such that $f = \bigcup_{x\in\mathbb{R}} [\bar{f}(x^-), \bar{f}(x^+)] \cap \mathbb{R}$.

One is then interested in the problem: find $u \in H^1(0,T;H_0^1(\Omega))$ such that

$$g \in f(\partial_t u) - \mathrm{Div}[a(x,u,\partial_t u)\nabla u + b(x,u,\partial_t u)\nabla\partial_t u] \quad \text{in } Q,$$

in the following sense: there exists $f^\#(\partial_t u) \in f(\partial_t u) \cap L^2(Q)$ such that

$$g = f^\#(\partial_t u) - \mathrm{Div}[a(x,u,\partial_t u)\nabla u + b(x,u,\partial_t u)\nabla\partial_t u] \quad \text{in } Q,$$

with the Cauchy condition: $u(0,\cdot) = u_0$ in Ω.

Let us denote by f^0 the principal section of f defined by $f^0(x) = \mathrm{proj}_{f(x)}0$ for any x.

One still assumes the hypotheses (H), but, the last part of H_2 becomes:

$\mathbf{H}'_2$: $f^0[L^2(Q)] \subset L^2(Q)$. Moreover, one considers one of the two hypotheses: either $\mathbf{H}'_{2,1}$: f is injective in the sense $f(x)\cap f(y) \neq \emptyset \Rightarrow x = y$; or $\mathbf{H}'_{2,2}$: a, b are Lipschitz-continuous with respect to u, v.

The aim of this section is to prove that.

Theorem 5. *There exist $u \in H^1(0,T,H_0^1(\Omega))$ and $f^\#(\partial_t u) \in f(\partial_t u) \cap L^2(Q)$ such that $u(0,.) = u_0$ and $\forall v \in H_0^1(\Omega)$, $t \in]0,T[$ a.e.,*

$$\int_\Omega \{f^\#(\partial_t u)v + [a(x,u,\partial_t u)\nabla u + b(x,u,\partial_t u)\nabla\partial_t u]\nabla v\}\mathrm{d}x = \langle g, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Proof. For any $\lambda > 0$, one considers the resolvent $J_\lambda = (Id + \lambda f)^{-1}$ and $f_\lambda = \frac{Id - J_\lambda}{\lambda}$ the Yosida approximation of f. f_λ is an increasing Lipschitz-continuous function, therefore, there exists a family u_λ of solutions to the pseudoparabolic problems (1) in the sense of Definition 1, regularized by replacing f by f_λ.

With the test-function $v = \partial_t u_\lambda$, one gets, for t a.e., that

$$\int_\Omega [f_\lambda(\partial_t u_\lambda) - f_\lambda(0)]\partial_t u_\lambda\,\mathrm{d}x + \beta\int_\Omega |\nabla\partial_t u_\lambda|^2\,\mathrm{d}x \le \|g\|_{H^{-1}(\Omega)}\|\partial_t u_\lambda\|_{H^1(\Omega)} + |f_\lambda(0)|meas(\Omega)\|\partial_t u_\lambda\|_{L^2(\Omega)} + M\|\nabla u_\lambda\|_{L^2(\Omega)}\|\nabla\partial_t u_\lambda\|_{L^2(\Omega)} \le \left(\sqrt{1+C_p^2}\|g\|_{H^{-1}(\Omega)} + C_p|f_\lambda(0)|meas(\Omega) + M\left\|\int_0^t \nabla\partial_t u_\lambda \mathrm{d}s\right\|_{L^2(\Omega)} + M\|\nabla u_0\|_{L^2(\Omega)}\right) \times \|\nabla\partial_t u_\lambda\|_{L^2(\Omega)},$$

where C_p denotes the Poincaré's constant. Then,

$$\int_\Omega [f_\lambda(\partial_t u_\lambda) - f_\lambda(0)]\partial_t u_\lambda\,\mathrm{d}x + \frac{\beta}{2}\int_\Omega |\nabla\partial_t u_\lambda|^2\,\mathrm{d}x \le \frac{2}{\beta}\left[(1+C_p^2)\|g\|^2_{H^{-1}(\Omega)} + C_p^2|f^0(0)|^2 meas(\Omega) + M^2T\int_0^t \|\nabla\partial_t u_\lambda\|^2_{L^2(\Omega)}\mathrm{d}s + M^2\|\nabla u_0\|^2_{L^2(\Omega)}\right],$$

and (u_λ) is bounded in $H^1(0,T;H_0^1(\Omega))$.

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In order to finish, one has to generalize the proof of Theorem 2 from (7). Let us denote by $(\partial_t u_{\lambda_t}(t))$ a subsequence that converges weakly in $H_0^1(\Omega)$, strongly in $L^2(\Omega)$ and a.e. in Ω to a function denoted by $\xi(t)$. Moreover, it can be assume that $(f_{\lambda_t}(\partial_t u_{\lambda_t}(t)))$ converges weakly in $L^2(\Omega)$ to an element $\chi(t)$. Indeed, otherwise there exists a subsequence, denoted in the same way for convenience, such that $(\|f_{\lambda_t}(\partial_t u_{\lambda_t}(t))\|_{L^2(\Omega)})$ is an increasing divergent sequence.

Since $(\partial_t u_{\lambda_t}(t))$ converges in $L^2(\Omega)$, there exists a subsequence $(\partial_t u_{\lambda_t'}(t))$ and $\varphi \in L^2(\Omega)$ such that $|\partial_t u_{\lambda_t'}(t)| \leq \varphi$ *a.e.* in Ω. Then, one gets a contradiction since, as f^0 is a nondecreasing function,

$$|f_{\lambda_t'}(\partial_t u_{\lambda_t'}(t))| \leq |f^0(\partial_t u_{\lambda_t'}(t))| \leq |f^0(\varphi)| + |f^0(-\varphi)| \in L^2(\Omega).$$

As, $[\partial_t u_{\lambda_t}(t), f_{\lambda_t}(\partial_t u_{\lambda_t}(t))] \in f_{\lambda_t}$, one gets that

$$[\partial_t u_{\lambda_t}(t) - \lambda_t f_{\lambda_t}(\partial_t u_{\lambda_t}(t)), f_{\lambda_t}(\partial_t u_{\lambda_t}(t))] \in f.$$

Since $\partial_t u_{\lambda_t}(t)$ converges in $L^2(\Omega)$ to $\xi(t)$ and $f_{\lambda_t}(\partial_t u_{\lambda_t}(t))$ converges weakly to $\chi(t)$ in $L^2(\Omega)$, $\partial_t u_{\lambda_t}(t) - \lambda_t f_{\lambda_t}(\partial_t u_{\lambda_t}(t))$ converges to $\xi(t)$ in $L^2(\Omega)$ and then $\int_\Omega(\partial_t u_{\lambda_t}(t) - \lambda_t f_{\lambda_t}(\partial_t u_{\lambda_t}(t)))f_{\lambda_t}(\partial_t u_{\lambda_t}(t))\,dx$ converges to $\int_\Omega(\xi(t))\chi(t)\,dx$. Since f is maximal monotone, thanks to Brézis [51] Prop. 2.5 p. 27, one has that $\chi(t) \in f(\xi(t))$.

Again, since f is monotone, Hypothesis H_2' allows us to conclude that $\xi(t)$ is unique and the proof of the existence of a solution still holds. □

4.0.4. Penalization of a constraint on $\partial_t u$

Assume that $f \subset \mathbb{R}^2$ is the graph of a maximal monotone operator as in the previous section. Assume that it conserves the bounded sets of $L^2(Q)$ and consider k_1 and k_2 such that $-\infty \leq k_1 < k_2 \leq +\infty$ with $0 \in [k_1, k_2]$.

Then, we are interested in finding a $u \in H^1(0, T; H_0^1(\Omega))$ solution to the penalized problem

$$g \in f(\partial_t u) - \Delta u - \mathrm{Div}[b(\partial_t u)\nabla\partial_t u] \quad \text{in } Q,$$

under the constraint $k_1 \leq \partial_t u \leq k_2$.

The aim of this section is to prove that.

Theorem 6. *There exist $u \in H^1(0, T, H_0^1(\Omega))$ with $k_1 \leq \partial_t u \leq k_2$ a.e. in Q and $f^\#(\partial_t u) \in f(\partial_t u) \cap L^2(Q)$ such that $u(0, .) = u_0$ and $\forall v \in H_0^1(\Omega)$ with $k_1 \leq v \leq k_2$ and $t \in]0, T[$ a.e.,*

$$\int_\Omega \{f^\#(\partial_t u)(v - \partial_t u) + [a(x, u, \partial_t u)\nabla u + b(x, u, \partial_t u)\nabla\partial_t u]\nabla(v - \partial_t u)\}dx = \langle g, v - \partial_t u\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Proof. Set $f_{k_1,k_2} = f + \partial I_{[k_1,k_2]}$ where $\partial I_{[k_1,k_2]}$ denotes the sub-differential of the indicatrice function of $[k_1, k_2]$. Set $\psi = \partial I_{[k_1,k_2]}$, ψ_λ its Yosida approximation and denote by u_λ the solution of the problem:

$$g \in (f + \psi_\lambda)(\partial_t u_\lambda) - \Delta u_\lambda - \mathrm{Div}[b(\partial_t u_\lambda)\nabla\partial_t u_\lambda] \quad \text{in } Q,$$

for the Cauchy condition

$$u_\lambda(0, \cdot) = u_0 \quad \text{dans } \Omega.$$

Thanks to the test-function $\partial_t u_\lambda$, one gets the estimate:

$$\int_\Omega [\psi_\lambda(\partial_t u_\lambda) - \psi_\lambda(0)]\partial_t u_\lambda\,dx + \int_\Omega [f^\#(\partial_t u_\lambda) - f^0(0)]\partial_t u_\lambda\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\lambda|^2\,dx + \frac{\beta}{2}\int_\Omega |\nabla\partial_t u_\lambda|^2\,dx \\ \leq \frac{2}{\beta}[(1 + C_p^2)\|g\|^2_{H^{-1}(\Omega)} + C_p^2\|f^0(0)\|^2_{L^2(\Omega)}].$$

Since f is monotone, (u_λ) is bounded in $H^1(0, T; H_0^1(\Omega))$.

Let us denote by u any limit-point in $H^1(0, T; H_0^1(\Omega))$ for the weak convergence, associated with a subsequence denoted in the same way. Then, it is possible to assume that $f^\#(\partial_t u_\lambda)$ converges weakly in $L^2(Q)$ to a given element χ.

Note moreover that

$$\frac{-1}{\lambda}\int_Q (k_1 - \partial_t u_\lambda)^+\partial_t u_\lambda\,dx + \frac{1}{\lambda}\int_Q (\partial_t u_\lambda - k_2)^+\partial_t u_\lambda\,dx \leq Cte.$$

Then, since $k_1 \leq 0 \leq k_2$, one has that

$$\int_Q (k_1 - \partial_t u_\lambda)^+(k_1 - \partial_t u_\lambda)\,dx + \int_Q (\partial_t u_\lambda - k_2)^+(\partial_t u_\lambda - k_2)\,dx \leq Cte\lambda,$$

and thus, $k_1 \leq \partial_t u \leq k_2$.

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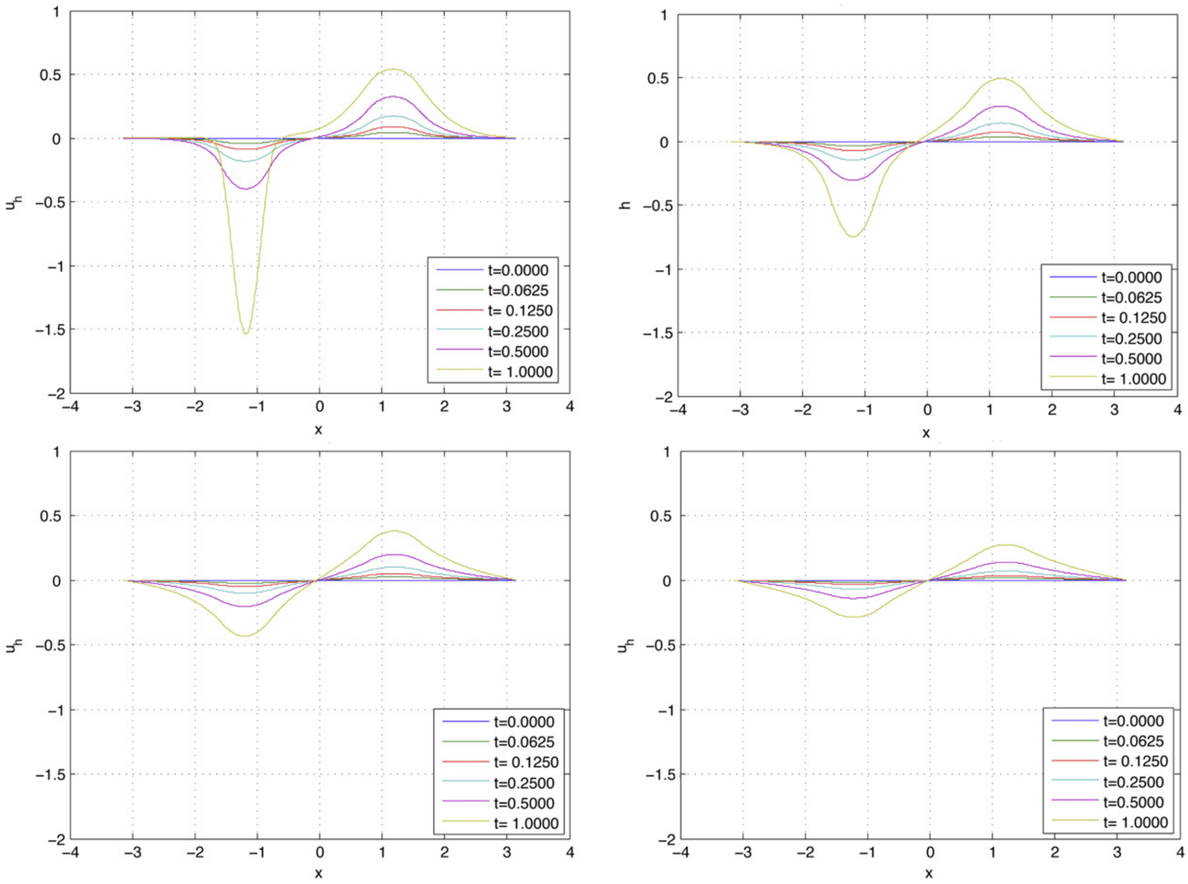

Fig. 1. Simulation when $\epsilon = 0.1, 0, 2, 0.5$ and 1 respectively.

Moreover, for any $v \in L^2(0,T;H_0^1(\Omega))$ such that $k_1 \le v \le k_2$, since ψ_λ is monotone, one gets that

$$\begin{aligned}&\int_\Omega \{f^\#(\partial_t u_\lambda)[v-\partial_t u_\lambda] + [\nabla u_\lambda + b(\partial_t u_\lambda)\nabla\partial_t u_\lambda]\nabla[v-\partial_t u_\lambda]\}\,\mathrm{d}x\\ &\quad = \langle g, v-\partial_t u_\lambda\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \int_\Omega [\psi_\lambda(\partial_t u_\lambda) - \psi_\lambda(v)][v-\partial_t u_\lambda]\,\mathrm{d}x\\ &\quad \ge \langle g, v-\partial_t u_\lambda - E\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.\end{aligned}$$

Since $b(\partial_t u_\lambda)\nabla\partial_t u_\lambda = \nabla B(\partial_t u_\lambda)$ where $B(x) = \int_0^x b(s)\mathrm{d}s$,

$$\begin{aligned}&\int_Q \chi v\,\mathrm{d}x + \int_Q \nabla u \nabla v\,\mathrm{d}x + \int_Q b(\partial_t u)\nabla\partial_t u\nabla v\,\mathrm{d}x \ge \int_0^T \langle g, v-\partial_t u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t\\ &\quad + \limsup_{\lambda\to 0}\left[\int_Q f^\#(\partial_t u_\lambda)[\partial_t u_\lambda]\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u_\lambda(T)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \int_Q \left|\nabla\int_0^{\partial_t u_\lambda}\sqrt{b}(s)\mathrm{d}s\right|^2\mathrm{d}x\right].\end{aligned}$$

Then, the lsc properties for the weak convergence yield

$$\begin{aligned}&\int_Q \chi v\,\mathrm{d}x + \int_Q \nabla u\nabla v\,\mathrm{d}x + \int_Q b(\partial_t u)\nabla\partial_t u\nabla v\,\mathrm{d}x\\ &\quad \ge \int_0^T \langle g, v-\partial_t u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\mathrm{d}t + \limsup_{\lambda\to 0}\left[\int_Q f^\#(\partial_t u_\lambda)[\partial_t u_\lambda]\,\mathrm{d}x\right]\\ &\qquad + \frac{1}{2}\int_\Omega |\nabla u(T)|^2\,\mathrm{d}x - \frac{1}{2}\int_\Omega |\nabla u_0|^2\,\mathrm{d}x + \int_Q \left|\nabla\int_0^{\partial_t u}\sqrt{b}(s)\mathrm{d}s\right|^2\mathrm{d}x.\end{aligned}$$

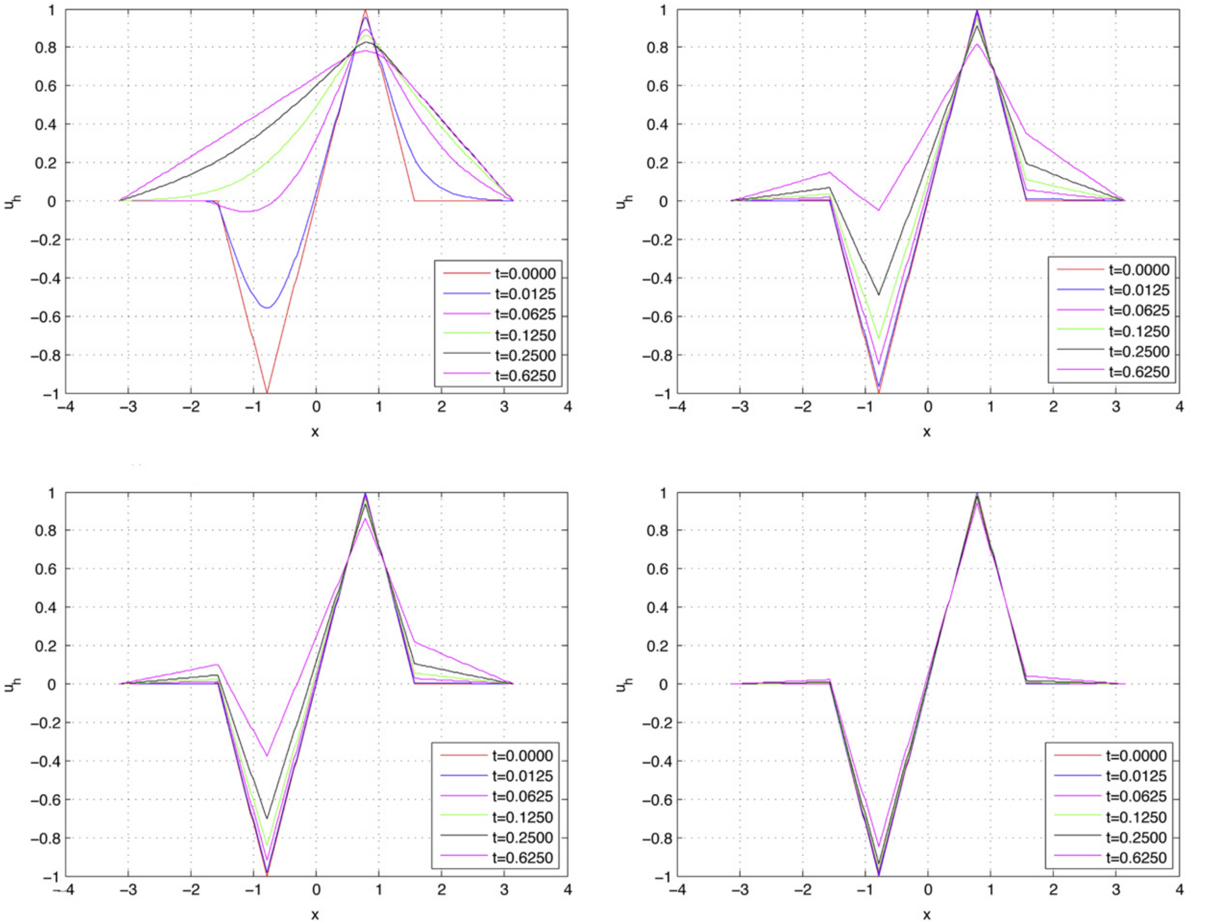

Fig. 2. Simulation when $\epsilon = 0, 0, 5, 1$ and 5 respectively and small time.

Thus,

$$\int_Q \chi v \,\mathrm{d}x + \int_Q \nabla u \nabla[v - \partial_t u] \,\mathrm{d}x + \int_Q b(\partial_t u)\nabla \partial_t u \nabla[v - \partial_t u] \,\mathrm{d}x$$
$$\geq \int_0^T \langle g, v - \partial_t u\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \mathrm{d}t + \limsup_{\lambda\to 0} \left[\int_Q f^{\#}(\partial_t u_\lambda)\partial_t u_\lambda \,\mathrm{d}x\right].$$

Then, for $v = \partial_t u$, one has that

$$\int_Q \chi \partial_t u \,\mathrm{d}x \geq \limsup_{\lambda\to 0} \left[\int_Q f^{\#}(\partial_t u_\lambda)\partial_t u_\lambda \,\mathrm{d}x\right].$$

Since, respectively, $f^{\#}(\partial_t u_\lambda)$ and $\partial_t u_\lambda$ converge weakly to χ and $\partial_t u$ in $L^2(Q)$, and since f is maximal monotone, Brezis [51] Prop. 2.5 p. 27 yields $\chi \in f(\partial_t u)$ and

$$\int_Q \chi \partial_t u \,\mathrm{d}x = \lim_{\lambda\to 0} \left[\int_Q f^{\#}(\partial_t u_\lambda)\partial_t u_\lambda \,\mathrm{d}x\right].$$

This last assertion allows us to prove the existence of a solution. □

5. Some numerical illustrations

In this section, one proposes two sets of numerical illustrations. The domain Ω is $]-\pi, \pi[$ and the curves are obtained by a standard P_1-finite element method in space and the time discretization introduced in the above main section.

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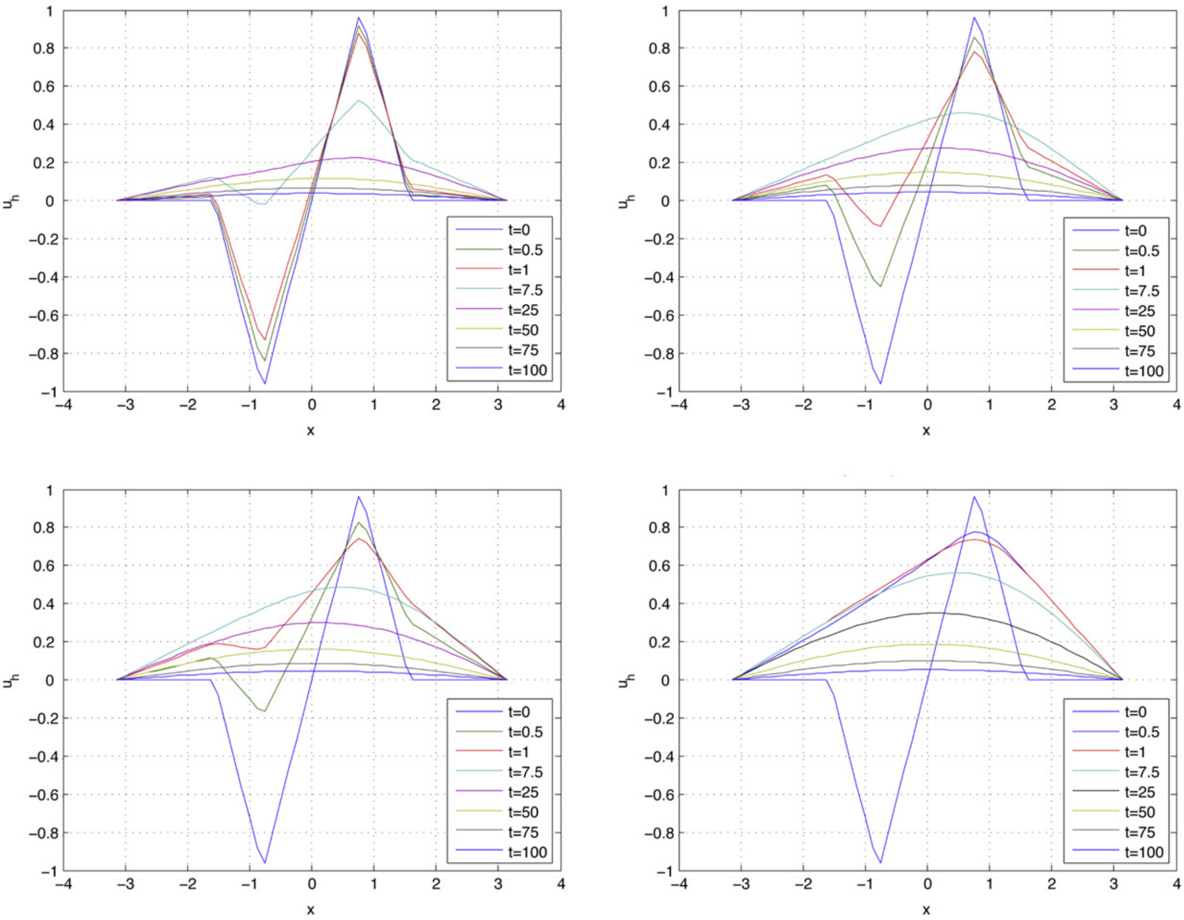

Fig. 3. Simulation when $\epsilon = 0, 0, 5, 1$ and 5 respectively and large time.

In the first one, the equation is $\partial_t u - \partial_x(\arctan(u)\partial_x u) - \epsilon\partial^3_{xxt}u = g$ where ϵ is a constant, $u_0 = 0$ and

$$g(t,x) = \begin{cases} 1 & \text{if } x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right], \\ -1 & \text{if } x \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right], \\ 0 & \text{else.} \end{cases}$$

Solutions are presented in Fig. 1 when $\epsilon = 0.1, 0.2, 0.5$ and 1. Note that $\arctan(u) < 0$ for negative values of u.

In a second set of curves, the equation is $f(\partial_t u) - \partial^2_{xx}u - \epsilon\partial^3_{xxt}u = 0$ where the initial condition is a continuous piecewise affine odd function and $f(r) = r/10$ if $r > 0$, $f(r) = 10r$ otherwise. We give in Fig. 2 the simulations for $\epsilon = 0, 0.5, 1$ and 5 for small values of the time t. Then, in Fig. 3, the same configuration is presented to illustrate an asymptotic behavior to 0 when t tends to infinity.

The first remark is that the pseudoparabolic perturbation slows down the evolution of the system and diffuses faster towards the boundaries of the interval.

A second one is that the initial condition fixes the regularity of the solution. Contrary to the heat equation, one has not a space-regularization when $\epsilon > 0$. One has even a transport of the singularities of the initial condition, in the sense that, for any positive t, $u(t) \notin H^2(-\pi, \pi)$ when $u_0 \notin H^2(-\pi, \pi)$.

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