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On Lewy–Stampacchia Inequalities for a Pseudomonotone Elliptic Bilateral Problem in Variable Exponent Sobolev Spaces

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Abstract. The aim of this work is to consider Lewy–Stampacchia inequalities for pseudomonotone elliptic operators in very general situations. This generalizes the results, and simplifies the proofs, proposed in the unilateral obstacle case, as well as the one in the bilateral case. By an ad hoc perturbation of the operator and a penalization of the constraint, one is able to reduce significantly the usual assumptions on the data and to consider a pseudomonotone elliptic operator defined on variable exponent Sobolev spaces.

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1. Introduction

In this paper, we are interested in proving Lewy–Stampacchia (LS) inequalities associated with constraints $\psi_1 \leq u$ and/or $u \leq \psi_2$, namely,

$$\begin{cases} \text{The right constraint LS inequality} \\ \quad -(A(\psi_2)+a_0(\psi_2)-f)^- \leq A(u)+a_0(u)-f, \\ \text{The left constraint LS inequality} \\ \quad A(u)+a_0(u)-f \leq (A(\psi_1)+a_0(\psi_1)-f)^+, \end{cases}$$

in the general framework of a nonlinear Leray–Lions pseudomonotone operator A, a monotone Nemitsky operator a_0 and a solution u to the variational inequality

$$u \in K, \qquad \langle A(u)+a_0(u), v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K,$$

where K is a closed convex subset from $W_0^{1,p(\cdot)}(\Omega)$ related to the constraints. We discuss also under which assumptions the two parts of the above Lewy–Stampacchia inequalities simultaneously hold.

Lewy and Stampacchia [10] proved the first inequality in the frame of superharmonic problems; then, many authors have been interested in the so-called Lewy–Stampacchia inequality associated with obstacle problems. Without trying to be exhaustive, let us cite the monograph of Rodrigues [16] and the papers of Mokrane and Murat [11] for pseudo-monotone elliptic problems, Mokrane and Vallet [13] in the context of Sobolev spaces with variable exponents, Rodrigues, Sanchón and Urbano [18] for proving the existence and uniqueness of an entropy solution to the obstacle problem for nonlinear elliptic equations with variable growth and L^1-data, Pinamonti and Valdinoci [15] in the framework of Heisenberg group, Servadei and Valdinoci [22] for nonlocal operators or Gigli and Mosconi [7] concerning an abstract presentation.

Concerning the bilateral problem, let us cite Mokrane and Murat [12] where the authors proved the existence of a solution satisfying LS inequality for a rather general Leray–Lions operator of second order by assuming the existence of a perturbed problem satisfying a uniqueness property. Let us also cite Rodrigues and Teymurazyan [19], where the authors proved LS inequality for the two obstacles problem in abstract form for a T-monotone operator in the frame of (generalized) Orlicz–Sobolev spaces. To the best of the authors' knowledge, there do not exist in the literature such general LS inequalities for pseudomonotone operators, nor generalizations of Mokrane and Murat work [12] to the case of variable exponent Sobolev spaces $W_0^{1,p(\cdot)}(\Omega)$.

In this paper, we propose such results by using a method of penalization, associated with a suitable perturbation of the operator as proposed e.g. by [9, p.102] and [3] for sub/super solutions to obstacle quasilinear elliptic problems. This perturbation is one of the main points of the proof: to make it possible and to reduce to the minimum the list of assumptions. We discuss also about additional conditions proposed, e.g., in [12] to derive a result in the general case.

The paper is organized in the following way: after giving the hypotheses and the main result (Theorem 2.3) in Sects. 2, 3 is devoted to the proof of this result. A first step is devoted to the existence of a solution to the penalized/perturbed problem associated with a parameter ϵ; then, some a priori estimates and passage to the limit with respect to η (subsequence of ϵ) are considered with regular non-negative elements g_1^+ and g_2^-, associated with decompositions of certain elements assumed to be in the order dual. We prove first the two parts of Lewy–Stampacchia inequality when g_1^+ and g_2^- are still regular; finally, the proof of Lewy–Stampacchia inequality is extended to the general case in the frame of unilateral problems. In Sect. 4, we discuss some additional assumptions to get the two parts of Lewy–Stampacchia inequality in the frame of bilateral problems simultaneously.

2. Notation, Hypotheses and Main Result

Let d be a natural number and $\Omega \subset \mathbb{R}^d$ a bounded domain with a Lipschitz boundary $\partial\Omega$. In the sequel, the exponent $p : \Omega \to [1, +\infty[$ is a measurable

function, and we set $p_- = ess\inf_{\Omega} p$ and $p^+ = ess\sup_{\Omega} p$. We assume also that p is a log-Holder continuous function (see, e.g., [4, p. 98]).
Consider the following variable exponent Sobolev spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$; one can consult [6] for the basic properties and some results concerning this type of spaces. The truncation nonlinearities are defined, for positive n and any real x, by $T_n(x) = \min[n, \max(-n, x)]$.
Denote, for given measurable functions $(\psi_i)_{i=1,2} : \Omega \to \bar{\mathbb{R}}$, by

$$K_{\psi_1} = \{u \in W_0^{1,p(\cdot)}(\Omega) : \psi_1 \leq u \,\text{a.e. in}\, \Omega\},$$
$$K^{\psi_2} = \{u \in W_0^{1,p(\cdot)}(\Omega) : u \leq \psi_2 \,\text{a.e. in}\, \Omega\},$$
$$\text{and}\, K(\psi_1, \psi_2) = K_{\psi_1} \cap K^{\psi_2} = \{u \in W_0^{1,p(\cdot)}(\Omega) : \psi_1 \leq u \leq \psi_2 \,\text{a.e. in}\, \Omega\}.$$

Assume that:

H_1 : A is a Leray–Lions pseudomonotone operator of the form

$$v \mapsto A(v) = -\text{div}\Big[a(x, v, \nabla v)\Big],$$

which acts from $W^{1,p(\cdot)}(\Omega)$ into $W^{-1,p'(\cdot)}(\Omega)$, where

$H_{1,1}$ $a : (x, u, \vec{\xi}) \in \Omega \times \mathbb{R} \times \mathbb{R}^d \mapsto a(x, u, \vec{\xi}) \in \mathbb{R}^d$ is a Carathéodory function on $\Omega \times \mathbb{R}^{d+1}$,

$H_{1,2}$ a is strictly monotone with respect to its last argument:

$$\forall x \in \Omega \text{ a.e.},\ u \in \mathbb{R},\ \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^d, \quad \vec{\xi} \neq \vec{\eta} \Rightarrow [a(x, u, \vec{\xi}) - a(x, u, \vec{\eta})] \cdot (\vec{\xi} - \vec{\eta}) > 0.$$

$H_{1,3}$: there exist constants $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{\gamma} \geq 0$, a function $\bar{h}$ in $L^1(\Omega)$ and a function $\bar{k}$ in $L^{p(\cdot)}(\Omega)$ and $1 \leq q(x), r(x) \leq q^+ < p^-$ two exponents such that, for a.e. $x \in \Omega$, $\forall u \in \mathbb{R}$, $\forall \vec{\xi} \in \mathbb{R}^d$,

$$a(x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha}|\vec{\xi}|^{p(x)} - \Big[\bar{\gamma}|u|^{q(x)} + |\bar{h}(x)|\Big], \tag{1}$$

$$|a(x, u, \vec{\xi})| \leq \bar{\beta}\Big[|\bar{k}(x)| + |u|^{\frac{r(x)}{p(x)}} + |\vec{\xi}|\Big]^{p(x)-1}. \tag{2}$$

H_2: a_0 is a nonlinear superposition operator acting form $L^{p(\cdot)}(\Omega)$ into its dual $L^{p'(\cdot)}(\Omega)$, which is defined by

$$a_0(u) = a_0(x, u), \tag{3}$$

where the function $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a monotone (non decreasing) Carathéodory function, i.e.,
$\forall s \in \mathbb{R}, x \mapsto a_0(x, s)$ is measurable, a.e. $x \in \Omega$, $s \mapsto a_0(x, s)$ is continuous, and a.e. $x \in \Omega$, $\forall s \in \mathbb{R}$, $\forall t \in \mathbb{R}$, $(a_0(x, s) - a_0(x, t))(s - t) \geq 0$.
We also assume that there exist a constant $\bar{\beta}_0 > 0$ and a function $\bar{k}_0$ in $L^{p(\cdot)}(\Omega)$, $q_1(x) \leq p(x)$ and a function $v \geq 0$ such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$ one has

$$\begin{cases} |a_0(x, s)| \leq \bar{\beta}_0(|\bar{k}_0(x)| + |s|)^{q_1(x)-1}, \\ \nabla q_1 \cdot \nabla v \geq 0, \quad |\nabla v| \neq 0 \quad in \quad \bar{\Omega}. \end{cases} \tag{4}$$

Remark 2.1. The second assumption on q_1 in (4) allows us to use Poincaré's inequality in modular form, i.e.,

$$\exists C_p > 0, \quad \forall v \in W_0^{1,q_1(x)}(\Omega), \quad \int_\Omega |v(x)|^{q_1(x)} \mathrm{d}x \leq C_p \int_\Omega |\nabla v(x)|^{q_1(x)} \mathrm{d}x,$$

(see [1, Th. 1]). Instead, we can also assume that $q_1(x) \leq p_-$ without assuming the second condition in (4) used to prove Lemma 3.2.

H_3 : $f \in W^{-1,p'(\cdot)}(\Omega)$, $(\psi_i)_{i=1,2} : \Omega \to \bar{\mathbb{R}}$ are measurable functions such that there exists v^* in $W_0^{1,p(\cdot)}(\Omega)$ such that $\psi_1 \leq v^* \leq \psi_2$.
H_4 : Define, for all $v \in W^{1,p(\cdot)}(\Omega)$, the operator B by $B(v) = A(v) + a_0(v) - f$ and denote the order dual space by

$$V^*_{p'(\cdot)} = (W^{-1,p'(\cdot)}(\Omega))^+ - (W^{-1,p'(\cdot)}(\Omega))^+.$$

$\mathbf{H}_{4,1}$: $\psi_1 \in W^{1,p(\cdot)}(\Omega)$, $B(\psi_1) \in V^*_{p'(\cdot)}$,
i.e., $B(\psi_1) = g_1^+ - g_1^-,\ g_1^+, g_1^- \in W^{-1,p'(\cdot)}(\Omega),\ g_1^+, g_1^- \geq 0.$
$\mathbf{H}_{4,2}$: $\psi_2 \in W^{1,p(\cdot)}(\Omega)$, $B(\psi_2) \in V^*_{p'(\cdot)}$,
i.e., $B(\psi_2) = g_2^+ - g_2^-,\ g_2^+, g_2^- \in W^{-1,p'(\cdot)}(\Omega),\ g_2^+, g_2^- \geq 0.$

Remark 2.2. Of course, H_3 and $\mathrm{H}_{4,i}$ $(i = 1, 2)$ yield $\psi_1 \leq 0$ and $\psi_2 \geq 0$ on $\partial\Omega$.
Reciprocally, by [4, Prop. 7.1.8 p. 244], assuming $\mathrm{H}_{4,1}$ with $\psi_1 \leq 0$ on $\partial\Omega$ yields ψ_1^+ belonging to K_{ψ_1}. Likewise, $\mathrm{H}_{4,2}$ with $\psi_2 \geq 0$ on $\partial\Omega$ yields $-\psi_2^-$ belonging to K^{ψ_2}. Then, assuming ψ_1 and ψ_2 in $W^{1,p(\cdot)}(\Omega)$ with $\psi_1 \leq \psi_2$ in Ω and $\psi_1 \leq 0 \leq \psi_2$ on $\partial\Omega$ ensures that $v^* = \psi_1^+ - \psi_2^-$ belongs to $K(\psi_1, \psi_2)$.

Our aim is to prove the following result and discuss under which assumptions the full Lewy–Stampacchia inequality (8) holds in the general framework.

Theorem 2.3. *Under the above assumptions* (H_1)*–*(H_3)*, there exists at least one solution* $u \in K(\psi_1, \psi_2)$ *which is a solution of the variational inequality*

$$\int_\Omega \Big[a(x, u, \nabla u)\nabla(v - u) + a_0(x, u)(v - u)\Big]\, dx \geq \langle f, v - u\rangle, \forall v \in K(\psi_1, \psi_2). \tag{5}$$

- *Assuming* $H_{4,1}$ *, there exists a solution of (5) such that*
$$B(u) \in V^*_{p'(\cdot)} \quad \text{and} \quad (B(u))^+ \leq (B(\psi_1))^+. \tag{6}$$
- *Assuming* $H_{4,2}$*, there exists a solution of (5) such that*
$$B(u) \in V^*_{p'(\cdot)} \quad \text{and} \quad (B(u))^- \leq (B(\psi_2))^-. \tag{7}$$
- *If* H_4 *holds true with* g_1^+ *and* g_2^- *in* $W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$*, then there exists a solution of (5) such that* $B(u) \in V^*_{p'(\cdot)}$ *and*

$$-(A(\psi_2) + a_0(\psi_2) - f)^- \leq A(u) + a_0(u) - f \leq (A(\psi_1) + a_0(\psi_1) - f)^+. \tag{8}$$

- *If H_4 holds true and if there exists a Nemitsky operator j on $\Omega \times \mathbb{R}$, satisfying H_2 like a_0, such that the solution $u \in K(\psi_1, \psi_2)$ to*

$$\int_\Omega a(x,u,\nabla u)\nabla(v-u)\,dx + \int_\Omega [a_0(x,u)+j(x,u)](v-u)\,dx \geq \langle f, v-u\rangle$$

for all $v \in K(\psi_1,\psi_2)$ is unique, then (8) is satisfied for any solution to (5).

3. Proof of Theorem 2.3

We will prove Theorem 2.3 with the following:

- The perturbed operator and some preliminary results.
- Proof of Lewy–Stampacchia inequality when g_1^+ and g_2^- are regular.
- Proof of Lewy Stampacchia inequality in the general case for unilateral problems.

3.1. The Perturbed Operator and Some Preliminary Results

Denote by $\tilde{a}(x,u,\xi) = a(x, \max(\psi_1, \min(u,\psi_2)), \xi)$ and $\tilde{A}$ is the operator associated with $\tilde{a}$.

Remark 3.1. We wish to draw the reader's attention to the fact that with the proposed perturbation: $\tilde{a}(x,u,\xi) = a(x, \max(\psi_1, min(u,\psi_2)), \xi)$, the operator is formally monotone and not pseudomonotone any more on the free set where the constraints are violated.

One will perform the proof in the bilateral case, but the unilateral cases correspond to $\psi_1 = -\infty$ or $\psi_2 = +\infty$.

Note that the above assumption H_1 still holds for $\tilde{a}$. Indeed,

$$\tilde{a}(x,u,\vec{\xi})\cdot\vec{\xi} \geq \bar{\alpha}|\vec{\xi}|^{p(x)} - \Big[\bar{\gamma}|\max(\psi_1, min(u,\psi_2))|^{q(x)} + |\bar{h}(x)|\Big], \tag{9}$$

$$|\tilde{a}(x,u,\xi)| \leq \bar{\beta}\Big[|\bar{k}(x)| + |\max(\psi_1, min(u,\psi_2))|^{\frac{r(x)}{p(x)}} + |\vec{\xi}|\Big]^{p(x)-1}. \tag{10}$$

Since by Assumption H_3, $|\max(\psi_1, min(u,\psi_2))|^{q(x)} \leq |u|^{q(x)} + |v^*|^{q(x)}$, one gets that

$$|\max(\psi_1, min(u,\psi_2))|^{\frac{r(x)}{p(x)}} \leq |u|^{\frac{r(x)}{p(x)}} + |v^*|^{\frac{r(x)}{p(x)}},$$

(1) and (2) are satisfied by replacing $\bar{h}$ by $\bar{h} + \bar{\gamma}|v^*|^{q(x)}$ and $\bar{k}$ by $\bar{k} + |v^*|^{\frac{r(x)}{p(x)}}$.

Lemma 3.2. *Assume H_1-H_2 and H_3. There exist positive constants δ and M such that, for any $u, v \in W_0^{1,p(\cdot)}(\Omega)$,*

$$\int_\Omega \tilde{a}(x,u,\nabla u)\nabla(u-v)\,dx + \int_\Omega a_0(x,u)(u-v)\,dx + M$$
$$\geq \frac{\bar{\alpha}}{2}\min\left(\|u\|^{p^+}_{W_0^{1,p(\cdot)}}, \|u\|^{p^-}_{W_0^{1,p(\cdot)}}\right) - \left[M\delta\int_\Omega |\nabla v(x)|^{p(x)}\,dx + M\int_\Omega |v(x)|^{q_1(x)}\,dx\right].$$

Proof. From [13, Lemma 4], there exist positive constants C, δ and C_1 such that, for any $u, v \in W_0^{1,p(\cdot)}(\Omega)$,

$$\int_\Omega \tilde{a}(x,u,\nabla u)\nabla(u-v)\mathrm{d}x + C \\ \geq (\bar{\alpha}-\delta)\int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - \delta\left[\|u\|^{q^+}_{W_0^{1,p(\cdot)}} + \|u\|^{r^+}_{W_0^{1,p(\cdot)}}\right] - C_1\delta\int_\Omega |\nabla v|^{p(x)}\mathrm{d}x.$$

On the other hand, for any $u, v \in W_0^{1,p(\cdot)}(\Omega)$, one has

$$a_0(x,u)(u-v) = (a_0(x,u) - a_0(x,v)(u-v) + a_0(x,v)(u-v) \geq a_0(x,v)(u-v),$$

since $|\int_\Omega a_0(x,v)v\mathrm{d}x| \leq C\int_\Omega |\bar{k}(x)|^{q_1(x)}\mathrm{d}x + C\int_\Omega |v(x)|^{q_1(x)}\mathrm{d}x$ where C is a positive constant. Thanks to Young's inequality and Remark 2.1, one has

$$|\int_\Omega a_0(x,v)u\mathrm{d}x| \leq C_\delta\int_\Omega (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)})\mathrm{d}x + \delta\int_\Omega |u(x)|^{q_1(x)}\mathrm{d}x, \\ \leq C_\delta\int_\Omega (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)})\mathrm{d}x + C_p\delta\int_\Omega |\nabla u(x)|^{q_1(x)}\mathrm{d}x,$$

where C_p is the positive constant of modular Poincaré's inequality. Using again Young's inequality, one has

$$|\int_\Omega a_0(x,v)u\mathrm{d}x| \leq C_\delta\int_\Omega (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)})\mathrm{d}x + \delta\int_\Omega |\nabla u(x)|^{p(x)}\mathrm{d}x + C',$$

where C_δ and C' are positive constants. Therefore,

$$\int_\Omega \tilde{a}(x,u,\nabla u)\nabla(u-v)\mathrm{d}x + \int_\Omega a_0(x,u)(u-v)\mathrm{d}x + C \\ \geq (\bar{\alpha}-\delta)\int_\Omega |\nabla u(x)|^{p(x)}\mathrm{d}x - \delta\left[\|u\|^{q^+}_{W_0^{1,p(\cdot)}} + \|u\|^{r^+}_{W_0^{1,p(\cdot)}}\right] \\ - C_1\delta\int_\Omega |\nabla v(x)|^{p(x)}\mathrm{d}x \\ - \left[C\int_\Omega |\bar{k}(x)|^{q_1(x)}\mathrm{d}x + C\int_\Omega |v(x)|^{q_1(x)}\mathrm{d}x\right] \\ - \left[C_\delta\int_\Omega (|\bar{k}(x)|^{q_1(x)} + |v(x)|^{q_1(x)})\mathrm{d}x + \delta\int_\Omega |\nabla u(x)|^{p(x)}\mathrm{d}x + C'\right].$$

Since $\bar{k} \in L^{p(\cdot)}(\Omega)$, by using Young's inequality and with a suitable choice of δ, there exists a constant $M > 0$ such that

$$\int_\Omega \tilde{a}(x,u,\nabla u)\nabla(u-v)\mathrm{d}x + \int_\Omega a_0(x,u)(u-v)\mathrm{d}x + M \\ \geq \frac{\bar{\alpha}}{2}\min\left(\|u\|^{p^+}_{W_0^{1,p(\cdot)}}, \|u\|^{p^-}_{W_0^{1,p(\cdot)}}\right) - M\left[\int_\Omega [\delta|\nabla v(x)|^{p(x)} + |v(x)|^{q_1(x)}\right]\mathrm{d}x.$$

□

Now, we consider the penalized problem.

Theorem 3.3. *Assume H_1-H_2 and H_3. Then for each $\epsilon > 0$, there exists at least one u_ϵ such that*

$$\begin{cases} u_\epsilon \in W_0^{1,p(\cdot)}(\Omega),\ (u_\epsilon - \psi_1)^- \in L^2(\Omega), \quad (u_\epsilon - \psi_2)^+ \in L^2(\Omega) \\ \int_\Omega \tilde{a}(x, u_\epsilon, \nabla u_\epsilon)\nabla v\, dx + \int_\Omega a_0(x, u_\epsilon) v\, dx - \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^- v\, dx \\ \quad + \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_2)^+ v\, dx = \langle f, v\rangle, \quad \forall v \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega). \end{cases} \tag{11}$$

Moreover, for all $v \in K(\psi_1, \psi_2)$, $(u_\epsilon - \psi_1)^-(u_\epsilon - v)$ and $(u_\epsilon - \psi_2)^+(u_\epsilon - v)$ are in $L^1(\Omega)$, and

$$\begin{aligned} &\int_\Omega \tilde{a}(x, u_\epsilon, \nabla u_\epsilon)\nabla(u_\epsilon - v)\, dx + \int_\Omega a_0(x, u_\epsilon)(u_\epsilon - v)\, dx \\ &\qquad -\frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^-(u_\epsilon - v)\, dx + \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_2)^+(u_\epsilon - v)\, dx \\ &\quad = \langle f, u_\epsilon - v\rangle. \end{aligned} \tag{12}$$

Proof. Note that $\tilde{A}$ is a coercive pseudomonotone operator [13, Rem. 1]. Then, denoting by $T_n = \max(-n, \min(\cdot, n))$ the truncation at height n, the operator

$$\mathrm{L} : w \mapsto -div[\tilde{a}(x, w, \nabla w)] + a_0(x, w) - \frac{1}{\epsilon}T_n(w - \psi_1)^- + \frac{1}{\epsilon}T_n(w - \psi_2)^+,$$

L is well defined from $W_0^{1,p(\cdot)}(\Omega)$ in $W^{-1,p'(\cdot)}(\Omega)$, and is a strongly continuous perturbation of $\tilde{A}$. For every $\epsilon > 0$, consider the problem

$$u_\epsilon^n \in W_0^{1,p(\cdot)}(\Omega), \quad \mathrm{L}u_\epsilon^n = f. \tag{13}$$

The existence of solution for (13) follows from [20, Th. 2.6]. With cosmetic updating of the proof [11, Th. 6.1], we get (11). The proof of (12) is similar to the one of [11, Prop. 6.1], and one just needs to verify that

$$(u_\epsilon - \psi_1)^-(u_\epsilon - v) \in L^1(\Omega) \quad \text{and} \quad (u_\epsilon - \psi_2)^+(u_\epsilon - v) \in L^1(\Omega).$$

We know that: $\forall v \in W_0^{1,p(\cdot)}(\Omega)$, $T_n(v)^+$ (resp. $T_n(v)^-$) belongs to $W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$, and $T_n(v)^+$ (resp. $T_n(v)^-$) tends strongly to v^+ (resp. v^-) in $W_0^{1,p(\cdot)}(\Omega)$.

Let $v \in K(\psi_1, \psi_2)$, consider $T_n(u_\epsilon - v)^-$ as test function in (11) and remark that:

$$\begin{cases} (u_\epsilon - \psi_2)^+ T_n(u_\epsilon - v)^- = 0 \quad \text{a.e. in } \Omega, \\ -\frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^- T_n(u_\epsilon - v)^- \mathrm{d}x \quad \text{is bounded independently of } n. \end{cases}$$

The monotone convergence theorem then implies that $(u_\epsilon - \psi_1)^-(u_\epsilon - v) \in L^1(\Omega)$ and

$$\begin{aligned} -\frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^- T_n(u_\epsilon - v)^- \mathrm{d}x &\longrightarrow -\frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^-(u_\epsilon - v)^- \mathrm{d}x \\ &= \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_1)^-(u_\epsilon - v)\mathrm{d}x. \end{aligned}$$

Using $T_n(u_\epsilon - v)^+$ as test function in (11), we remark similarly that:

$$\begin{cases} (u_\epsilon - \psi_1)^- T_n(u_\epsilon - v)^+ = 0 \quad \text{a.e. in}\,\Omega, \\ \dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_\epsilon - \psi_2)^+ T_n(u_\epsilon - v)^+ \mathrm{d}x \quad \text{is bounded independently of}\, n. \end{cases}$$

Again, by monotone convergence theorem, $(u_\epsilon - \psi_2)^+(u_\epsilon - v) \in L^1(\Omega)$ and

$$\begin{aligned} \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_2)^+ T_n(u_\epsilon - v)^+ \mathrm{d}x &\longrightarrow -\frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_2)^+ (u_\epsilon - v)^+ \mathrm{d}x \\ &= \frac{1}{\epsilon}\int_\Omega (u_\epsilon - \psi_2)^+ (u_\epsilon - v) \mathrm{d}x. \end{aligned}$$

□

For all $v \in K(\psi_1, \psi_2)$, we have

$$\begin{cases} -\dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_\epsilon - \psi_1)^- (u_\epsilon - v)\mathrm{d}x = \dfrac{1}{\epsilon}\displaystyle\int_\Omega |(u_\epsilon - \psi_1)^-|^2 + (u_\epsilon - \psi_1)^-(v - \psi_1)\mathrm{d}x, \\ \quad \geq \dfrac{1}{\epsilon}\displaystyle\int_\Omega |(u_\epsilon - \psi_1)^-|^2 \mathrm{d}x \geq 0, \\ \dfrac{1}{\epsilon}\displaystyle\int_\Omega (u_\epsilon - \psi_2)^+ (u_\epsilon - v)\mathrm{d}x = \dfrac{1}{\epsilon}\displaystyle\int_\Omega |(u_\epsilon - \psi_2)^+|^2 + (u_\epsilon - \psi_2)^+(\psi_2 - v)\mathrm{d}x, \\ \quad \geq \dfrac{1}{\epsilon}\displaystyle\int_\Omega |(u_\epsilon - \psi_2)^+|^2 \mathrm{d}x \geq 0. \end{cases} \tag{14}$$

Thanks to Lemma 3.2 and previous calculations, there exists a constant $C > 0$ independent of ϵ such that

$$\begin{cases} \|u_\epsilon\|_{W_0^{1,p(\cdot)}(\Omega)} + \|\tilde{a}(x, u_\epsilon, \nabla u_\epsilon)\|_{(L^{p'(\cdot)}(\Omega))^d} \leq C, \\ \|(u_\epsilon - \psi_1)^-\|^2_{L^2(\Omega)} + \|(u_\epsilon - \psi_2)^+\|^2_{L^2(\Omega)} \leq C\epsilon. \end{cases} \tag{15}$$

Thus, we can extract a subsequence, denoted by η, such that

$$u_\eta \rightharpoonup u \quad in\, W_0^{1,p(\cdot)}(\Omega) \quad \text{and a.e. in } \Omega, \tag{16}$$

$$\tilde{a}(x, u_\eta, \nabla u_\eta) \rightharpoonup \chi \quad \text{in}\, (L^{p'(\cdot)}(\Omega))^d. \tag{17}$$

In view of (15), we get

$$u \in K(\psi_1, \psi_2). \tag{18}$$

By (16), the Sobolev embedding theorem and the growth condition (4), we have

$$a_0(x, u_\eta) \to a_0(x, u) \quad \text{in}\, L^{p'(\cdot)}(\Omega). \tag{19}$$

Using (14) and passing to the limit in (12), we obtain, for all $v \in K(\psi_1, \psi_2)$,

$$\begin{aligned} &\limsup_\eta \int_\Omega \tilde{a}(x, u_\eta, \nabla u_\eta)\nabla u_\eta \mathrm{d}x - \int_\Omega \chi \nabla v \mathrm{d}x \\ &\quad + \int_\Omega a_0(x, u)(u - v)\mathrm{d}x \leq \langle f, u - v\rangle. \end{aligned} \tag{20}$$

Using (18) and taking $v = u$, we get

$$\limsup_\eta \int_\Omega \tilde{a}(x, u_\eta, \nabla u_\eta)\nabla u_\eta \mathrm{d}x \leq \int_\Omega \chi \nabla u \mathrm{d}x. \tag{21}$$

Since $v \mapsto \tilde{A}(v) = -div[\tilde{a}(x,v,\nabla v)]$ is a pseudomonotone operator, one has $div\chi = div[\tilde{a}(x,u,\nabla u)]$ and the first part of the following result holds.

Proposition 3.4.

$$\int_\Omega \tilde{a}(x,u_\eta,\nabla u_\eta)\nabla u_\eta dx \to \int_\Omega \tilde{a}(x,u,\nabla u)\nabla u dx. \tag{22}$$

Moreover, $\chi = \tilde{a}(x,u,\nabla u)$.

Indeed, similarly to the proof of [2, Lemma 1], we get that $\nabla u_\eta \to \nabla u$ in measure. Therefore, there exists a subsequence denoted by the same way such that $\nabla u_\eta \to \nabla u$ a.e. Using the continuity of $\tilde{a}$ with respect to its second and third arguments, we get

$$\tilde{a}(x,u_\eta,\nabla u_\eta) \to \tilde{a}(x,u,\nabla u) \quad \text{a.e. in } \Omega;$$

therefore, $\chi = \tilde{a}(x,u,\nabla u)$. Thanks to the previous calculations, we deduce

Proposition 3.5. *Assume H_1-H_3 hold true. Then there exists at least a solution $u \in K(\psi_1,\psi_2)$ to the variational inequality*

$$\int_\Omega a(x,u,\nabla u)\nabla(v-u)dx + \int_\Omega a_0(x,u)(v-u)dx \geq \langle f, v-u\rangle, \forall v \in K(\psi_1,\psi_2).$$

Note that the cases corresponding to K_{ψ_1} and K^{ψ_2} are similar by assuming formally $\psi_2 = +\infty$ or $\psi_1 = -\infty$.

3.2. Proof of Lewy–Stampacchia's Inequality with Regular g_1^+ and g_2^-.

Define $\mu_\eta^1 = \dfrac{1}{\eta}(u_\eta - \psi_1)^- \geq 0$, $\mu_\eta^2 = \dfrac{1}{\eta}(u_\eta - \psi_2)^+ \geq 0$. We have (see Theorem 3.3)

$$\mu_\eta^1 \in L^2(\Omega), \quad \mu_\eta^2 \in L^2(\Omega).$$

Take $v \in \mathcal{C}_c^\infty(\Omega)$ as test function in (11) and $\epsilon = \eta$, we get, thanks to Proposition 3.4,

$$\mu_\eta^1 - \mu_\eta^2 \rightharpoonup -div(\tilde{a}(x,u,\nabla u) + a_0(x,u) - f \quad \text{in}\, W^{-1,p'(\cdot)}(\Omega). \tag{23}$$

In this subsection, we consider the subsequence $(u_\eta)_\eta$ which satisfies the penalized problem (11). Thanks to the selected test function, we prove the two parts of Lewy–Stampacchia inequality independently and we get at the limit the two parts of Lewy–Stampacchia inequality since $(u_\eta)_\eta$ converges to the same limit u.

3.2.1. First Lewy–Stampacchia Inequality.

In this subsection, we assume that $0 \leq g_1^+ = (B(\psi_1))^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$. Denote by

$$z_\eta = g_1^+ - \frac{1}{\eta}(u_\eta - \psi_1)^-.$$

Note that $(u_\eta - \psi_1)^- \in W_0^{1,p(\cdot)}(\Omega)$ since $\psi_1 \leq 0$ on $\partial\Omega$, and from Theorem 3.3, $(u_\eta - \psi_1)^- \in L^2(\Omega)$. Therefore, $z_\eta \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$.

Lemma 3.6. *There exists a constant C, such that for any η,*

$$\int_\Omega \Big| \big[\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_1,\nabla\psi_1)\big]\cdot\nabla(u_\eta-\psi_1)^-\Big| dx \le C\eta\|g_1^+\|^2_{L^2(\Omega)},$$

$$\frac{1}{\eta}\|(u_\eta-\psi_1)^-\|^2_{L^2(\Omega)} \le C\eta\|g_1^+\|^2_{L^2(\Omega)}.$$

Proof. With the admissible test function $v = -(u_\eta-\psi_1)^-$ in (11), one has

$$\begin{cases} -\displaystyle\int_\Omega \big[\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_1,\nabla\psi_1)\big]\nabla(u_\eta-\psi_1)^-\mathrm{d}x \\ \quad -\displaystyle\int_\Omega \big[a_0(x,u_\eta)-a_0(x,\psi_1)\big](u_\eta-\psi_1)^-\mathrm{d}x + \frac{1}{\eta}\int_\Omega |(u_\eta-\psi_1)^-|^2\mathrm{d}x \\ \quad\quad -\displaystyle\frac{1}{\eta}\int_\Omega (u_\eta-\psi_2)^+(u_\eta-\psi_1)^-\mathrm{d}x \\ = -\langle f + div(\tilde{a}(x,\psi_1,\nabla\psi_1)) - a_0(x,\psi_1), (u_\eta-\psi_1)^-\rangle \\ = \langle g_1^+ - g_1^-, (u_\eta-\psi_1)^-\rangle \le 2\eta\|g_1^+\|^2_{L^2(\Omega)} + \displaystyle\frac{1}{2\eta}\int_\Omega |(u_\eta-\psi_1)^-|^2\mathrm{d}x. \end{cases}$$

Since $\psi_1 \le \psi_2$ in Ω, then $(u_\eta-\psi_2)^+(u_\eta-\psi_1)^- = 0$ a.e. in Ω. Therefore,

$$-\frac{1}{\eta}\int_\Omega (u_\eta-\psi_2)^+(u_\eta-\psi_1)^-\mathrm{d}x = 0.$$

Since $-(u_\eta-\psi_1)^- = (u_\eta-\psi_1)1_{\{u_\eta<\psi_1\}}$ and a_0 is non-decreasing with respect to its second argument, one has

$$-\int_\Omega \big[a_0(x,u_\eta)-a_0(x,\psi_1)\big](u_\eta-\psi_1)^-\mathrm{d}x \ge 0.$$

Then,

$$\int_{\{u-\psi_1<0\}} \Big[\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_1,\nabla\psi_1)\Big]\cdot\nabla(u_\eta-\psi_1)\mathrm{d}x + \frac{1}{2\eta}\int_\Omega |(u_\eta-\psi_1)^-|^2\mathrm{d}x \le 2\eta\|g_1^+\|^2_{L^2(\Omega)}.$$

Since $\tilde{a}(x,u_\eta,\nabla u_\eta) = a(x,\psi_1,\nabla u_\eta)$ in $\{u_\eta<\psi_1\}$, one gets that

$$\int_\Omega \Big|\big[\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_1,\nabla\psi_1)\big]\cdot\nabla(u_\eta-\psi_1)^-\Big|\mathrm{d}x + \frac{1}{2\eta}\int_\Omega |(u_\eta-\psi_1)^-|^2\mathrm{d}x \le 2\eta\|g_1^+\|^2_{L^2(\Omega)}.$$

□

We have $\tilde{A}(u_\eta) - A(\psi_1) + a_0(u_\eta) - a_0(\psi_1) + z_\eta + \frac{1}{\eta}(u_\eta-\psi_2)^+ = g_1^-$, where $z_\eta = g_1^+ - \frac{1}{\eta}(u_\eta-\psi_1)^-$. Our aim is to prove the strong convergence of z_η^- to 0 in $L^2(\Omega)$. Using $-z_\eta^-$ as test function in (11), one has

$$-\int_\Omega \big[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\big]\nabla z_\eta^-\mathrm{d}x-\int_\Omega \big[a_0(x,u_\eta)-a_0(x,\psi_1)\big]z_\eta^-\mathrm{d}x$$
$$+\int_\Omega |z_\eta^-|^2\mathrm{d}x-\frac{1}{\eta}\int_\Omega (u_\eta-\psi_2)^+z_\eta^-\mathrm{d}x=\langle g_1^-,-z_\eta^-\rangle\le 0.$$

Denote by $\mathcal{B}$ the set $\{g_1^+-\frac{1}{\eta}[(u_\eta-\psi_1)^-]<0\}=\{z_\eta<0\}$. On the one hand, since $u_\eta<\psi_1$ on $\mathcal{B}$, one has

$$-\frac{1}{\eta}\int_\Omega (u_\eta-\psi_2)^+z_\eta^-\mathrm{d}x=0,\quad -\int_\Omega \big[a_0(x,u_\eta)-a_0(x,\psi_1)\big]z_\eta^-\mathrm{d}x\ge 0.$$

On the other hand, note that,

$$\begin{aligned}&-\int_\Omega \big[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\big]\cdot\nabla z_\eta^-\mathrm{d}x\\ &=\int_\Omega 1_{\mathcal{B}}\left[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\right]\nabla\left[g_1^+-\frac{1}{\eta}[(u_\eta-\psi_1)^-]\right]\mathrm{d}x\\ &=\int_\Omega 1_{\mathcal{B}}\left[\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\right]\nabla\left[g_1^+-\frac{1}{\eta}[(u_\eta-\psi_1)^-]\right]\mathrm{d}x,\end{aligned}$$

since in this situation the integration holds in the set $\{u_\eta<\psi_1\}$. Thus,

$$\begin{cases}\Big[\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big]\nabla[g_1^+-\frac{1}{\eta}[(u_\eta-\psi_1)^-]]\\ \ge\frac{1}{\eta}\Big[\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big]\nabla(u_\eta-\psi_1)\\ \quad -\Big|\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big||\nabla g_1^+|\\ \ge-\Big|\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big||\nabla g_1^+|.\end{cases}$$

Thanks to the first estimate of Lemma 3.6,

$$\Big[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big]\nabla(u_\eta-\psi_1)^-\to 0 \text{ in } L^1(\Omega).$$

Then, by assumptions $H_{1,1}$ to $H_{1,3}$, up to a subsequence denoted in the same way, one gets that

$$\nabla(u_\eta-\psi_1)^-(x)\to 0\quad\text{a.e. in}\,\Omega. \tag{24}$$

Indeed, up to a subsequence denoted in the same way, u_η converges to u a.e. in Ω with $u\ge\psi_1$ a.e. and

$$\left|\Big[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big]\cdot\nabla(u_\eta-\psi_1)^-\right|\to 0\quad\text{a.e. in}\,\Omega. \tag{25}$$

Consider x such that the above limit (25) holds.

Thanks to Young's inequality, there exist $\delta>0$ and $C>0$ such that

$$\begin{aligned}|\tilde{a}(x,\psi_1,\nabla u_\eta)\cdot\nabla\psi_1|\le C(p^+,p_-,\delta,\bar{\beta})|\nabla\psi_1|^{p(x)}\\ +\delta 3^{p^+-1}(|\bar{k}|+|\psi_1|^{r(x)}+|\nabla u_\eta|)\end{aligned}$$

with suitable choice of δ, one gets

$$|\tilde{a}(x,\psi_1,\nabla u_\eta)\cdot\nabla\psi_1|\le C(p^+,p_-,r,\psi_1,\nabla\psi_1,\bar{k},\bar{\beta})+\frac{\bar{\alpha}}{2}|\nabla u_\eta|.$$

Since $\quad-\tilde{a}(x,\psi_1,\nabla u_\eta)\cdot\nabla(u_\eta-\psi_1)^-$

$$\ge\Big[\bar{\alpha}|\nabla u_\eta|^{p(x)}-\bar{\gamma}|\psi_1|^{q(x)}-|\bar{h}|-\tilde{a}(x,\psi_1,\nabla u_\eta)\cdot\nabla\psi_1\Big]1_{\{u_\eta<\psi_1\}},$$

$$\begin{aligned}&\text{then}\quad -\tilde a(x,u_\eta,\nabla u_\eta)\cdot\nabla(u_\eta-\psi_1)^-\\&\quad\geq\Big[\frac{\bar\alpha}{2}|\nabla u_\eta|^{p(x)}-C(p^+,p_-,r,\psi_1,\nabla\psi_1,\bar k,\bar h,\bar\beta)\Big]1_{\{u_\eta<\psi_1\}}.\\&\text{We have}\quad |\tilde a(x,\psi_1,\nabla\psi_1)\cdot\nabla(u_\eta-\psi_1)^-|\\&\quad\leq\bar\beta\Big[|\bar k|+|\psi_1|^{\frac{r(x)}{p(x)}}+|\nabla\psi_1|\Big]^{p(x)-1}\Big[|\nabla u_\eta|+|\nabla\psi_1|\Big]1_{\{u_\eta<\psi_1\}}.\end{aligned}$$

Thanks to Young's inequality, there exist $\delta>0$ and $C>0$ such that

$$\begin{aligned}&|\tilde a(x,\psi_1,\nabla\psi_1)\cdot\nabla(u_\eta-\psi_1)^-|\\&\quad\leq\Big[C(p^+,p_-,\delta,\bar\beta)\Big[|\bar k|+|\psi_1|^{\frac{r(x)}{p(x)}}+|\nabla\psi_1|\Big]^{p(x)}\\&\qquad+\delta\Big[|\nabla u_\eta|+|\nabla\psi_1|\Big]^{p(x)}\Big]1_{\{u_\eta<\psi_1\}}.\end{aligned}$$

With suitable choice of δ, one gets

$$\begin{aligned}&|\tilde a(x,\psi_1,\nabla\psi_1)\cdot\nabla(u_\eta-\psi_1)^-|\\&\quad\leq\Big[C(p^+,p_-,r,\psi_1,\nabla\psi_1,\bar k,\bar\beta)+\frac{\bar\alpha}{4}|\nabla u_\eta|^{p(x)}\Big]1_{\{u_\eta<\psi_1\}}.\\&\text{Therefore,}\quad\Big(-\Big[\tilde a(x,u_\eta,\nabla u_\eta)-\tilde a(x,\psi_1,\nabla\psi_1)\Big]\Big)\cdot\nabla(u_\eta-\psi_1)^-\\&\quad\geq\Big[\frac{\bar\alpha}{4}|\nabla u_\eta|^{p(x)}-C(p^+,p_-,r,\psi_1,\nabla\psi_1,\bar k,\bar h,\bar\beta)\Big]1_{\{u_\eta<\psi_1\}}.\end{aligned}$$

Using (25), one gets that $(\nabla u_\eta 1_{\{u_\eta<\psi_1\}})_\eta$ is a bounded sequence in $\mathbb{R}^d$. Thus, $(\nabla(u_\eta-\psi_1)^-(x))_\eta$ is a bounded sequence in $\mathbb{R}^d$.

Since $\nabla(u_\eta-\psi_1)^-(x)=-\nabla(u_\eta-\psi_1)(x)1_{\{u_\eta<\psi_1\}}(x)$, it converges to 0 if $u(x)>\psi_1(x)$. Else, at the limit, one has that $u(x)=\psi_1(x)$.

If one assumes that $\nabla(u_\eta-\psi_1)^-(x)$ is not converging to 0, then there exists a subsequence η' (depending on x) such that $\|\nabla(u_{\eta'}-\psi_1)^-(x)\|\geq\delta>0$ for a positive δ. Then, necessarily $-\nabla(u_{\eta'}-\psi_1)^-(x)=\nabla(u_{\eta'}-\psi)(x)$ and, since it is a bounded sequence in $\mathbb{R}^d$, there exists $\vec\xi\in\mathbb{R}^d$ and a new subsequence still labeled η' such that $\nabla u_{\eta'}(x)$ converges to $\vec\xi$, with the additional information: $\|\vec\xi-\nabla\psi_1(x)\|\geq\delta>0$. Therefore, since $\vec\xi\neq\nabla\psi_1(x)$

$$\begin{aligned}&\Big[\tilde a(x,u_{\eta'}(x),\nabla u_{\eta'}(x))-\tilde a(x,\psi_1(x),\nabla\psi_1(x))\Big]\nabla(u_{\eta'}-\psi_1)^-(x)\\&\quad=-\Big[\tilde a(x,\psi_1(x),\nabla u_{\eta'}(x))-\tilde a(x,\psi_1(x),\nabla\psi_1(x))\Big]\nabla(u_{\eta'}-\psi_1)(x),\end{aligned}$$

the last term converges to $-\Big[\tilde a(x,\psi_1(x),\vec\xi)-\tilde a(x,\psi_1(x),\nabla\psi_1(x))\Big][\vec\xi-\nabla\psi_1(x)]<0$. But, this is in contradiction with the convergence of the same sequence to 0 and (24) holds. Note that for x a.e. in Ω,

$$\begin{cases}\Big[\tilde a(x,u_\eta(x),\nabla u_\eta(x))-\tilde a(x,\psi_1(x),\nabla\psi_1(x))\Big]1_{\{u_\eta<\psi_1\}}\\\quad=\Big[\tilde a(x,\psi_1(x),\nabla u_\eta 1_{\{u_\eta<\psi_1\}}(x))-\tilde a(x,\psi_1(x),\nabla\psi_1 1_{\{u_\eta<\psi_1\}}(x))\Big]\end{cases}$$

and $\nabla u_\eta 1_{\{u_\eta<\psi_1\}}(x)-\nabla\psi_1 1_{\{u_\eta<\psi_1\}}(x)$ converges to 0. Then

$$\Big[\tilde a(x,u_\eta,\nabla u_\eta)-\tilde a(x,\psi_1,\nabla\psi_1)\Big]1_{\{u_\eta<\psi_1\}}\quad\text{converges a.e. to}\,0.$$

Lemma 3.7. *Let (u_n) be a bounded sequence in $L^{p(x)}(\Omega)$ and assume moreover that u_n converges a.e. to u. Then, $u_n \rightharpoonup u \quad in\, L^{p(x)}(\Omega)$.*

Indeed, the result is true when p is a constant greater than 1. Then, (u_n) is a bounded sequence in $L^{p_-}(\Omega)$ and

$$u_n \rightharpoonup u \quad \text{in}\, L^{p_-}(\Omega) \quad \text{and in}\, \mathcal{D}'(\Omega).$$

Since $L^{p(x)}(\Omega)$ is a reflexive Banach space, there exists a subsequence denoted by the same way and $v \in L^{p(x)}(\Omega)$ such that

$$u_n \rightharpoonup v \quad \text{in}\, L^{p(x)}(\Omega) \quad \text{and in}\, \mathcal{D}'(\Omega).$$

The uniqueness of the limit, in $\mathcal{D}'(\Omega)$, ensures the proof of the lemma.

Since $(\Big[\tilde{a}(x,\psi_1,\nabla u_\eta)-\tilde{a}(x,\psi_1,\nabla\psi_1)\Big]1_{\{u_\eta<\psi_1\}})$ is bounded in $L^{p'(x)}(\Omega)$, by Lemma 3.7 it converges weakly to 0 in $L^{p'(x)}(\Omega)$ and

$$\int_\Omega 1_{\{g_1^+ - \frac{1}{\eta}[(u_\eta-\psi_1)^-]<0\}}\Big|\tilde{a}(x,\psi_1,\nabla u_\eta) - \tilde{a}(x,\psi_1,\nabla\psi_1)\Big||\nabla g_1^+|\mathrm{d}x \to 0.$$

As a conclusion, z_η^- converges to 0 in $L^2(\Omega)$. Since $z_\eta = g_1^+ - \mu_\eta^1$ and $\mu_\eta^1 \geq 0$, this implies that

$$0 \leq \mu_\eta^1 \leq g_1^+ + z_\eta^-.$$

Since $z_\eta^- \to 0$ in $L^2(\Omega)$, μ_η^1 is bounded in $L^2(\Omega)$, by extracting a subsequence, there exists a non-negative function μ_1 such that

$$\mu_\eta^1 \rightharpoonup \mu^1 \quad in\, L^2(\Omega) \quad \text{resp. in } \mathcal{D}'(\Omega) \,\text{and} \quad 0 \leq \mu^1 \leq g_1^+. \tag{26}$$

But (23) implies that there exists a measure μ^2 such that

$$\begin{aligned} \mu_\eta^2 \rightharpoonup \mu^2 \quad in\, \mathcal{D}'(\Omega) \quad \text{and} \quad \mu^2 \geq 0, \quad \mu^1 - \mu^2 \\ = -div[a(x,u,\nabla u)] + a_0(x,u) - f. \end{aligned}$$

Since $g_1^+ \in L^\infty(\Omega)$ then $\mu^1 \in L^\infty(\Omega)$ and therefore μ^1 belongs to $W^{-1,p'(\cdot)}(\Omega)$ and μ^2 belongs to $W^{-1,p'(\cdot)}(\Omega)$. We have proved $B(u) \in V^*_{p'(\cdot)}$ and

$$(B(u))^+ = \mu^1 \leq g_1^+ = (B(\psi_1))^+$$

which implies that $B(u) \leq (B(\psi_1))^+$.

Remark 3.8. We can prove the above Lewy–Stampacchia inequality without proving $B(u) \in V^*_{p'(\cdot)}$ as following, $\quad z_\eta = g_1^+ - \frac{1}{\eta}[(u_\eta - \psi_1)^-]$

$$\begin{aligned} &\Rightarrow z_\eta^+ - \text{div}[\tilde{a}(\cdot,u_\eta,\nabla u_\eta)] + a_0(\cdot,u_\eta) + \frac{1}{\eta}(u_\eta - \psi_2)^+ - f = g_1^+ + z_\eta^- \\ &\Rightarrow - \text{div}[\tilde{a}(\cdot,u,\nabla u)] + a_0(\cdot,u) - f \leq g_1^+. \end{aligned}$$

Since $u \in K(\psi_1,\psi_2)$, then $\tilde{a}(\cdot,u,\nabla u) = a(\cdot,u,\nabla u)$. Therefore

$$-\text{div}[a(\cdot,u,\nabla u)] + a_0(\cdot,u) - f \leq g_1^+.$$

3.2.2. Second Lewy–Stampacchia Inequality. In this subsection, we assume that $0 \leq g_2^- = (B(\psi_2))^- \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

We just give a sketch of the proof of this second Lewy–Stampacchia inequality since this will be done similarly to the one proposed in Sect. 3.2.1. By considering the same subsequence (u_η) used in the Sect. 3.2.1, which satisfies the penalized problem (11), denote by

$$z_\eta = \frac{1}{\eta}(u_\eta - \psi_2)^+ - g_2^-.$$

Note that $(u_\eta - \psi_2)^+ \in W_0^{1,p(\cdot)}(\Omega)$ since $\psi_2 \geq 0$ on $\partial\Omega$, and from Theorem 3.3, $(u_\eta - \psi_2)^+ \in L^2(\Omega)$. Therefore $z_\eta \in W_0^{1,p(\cdot)}(\Omega) \cap L^2(\Omega)$.

Lemma 3.9. *There exists a constant C, such that for any η,*

$$\int_\Omega \Big| [\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_2,\nabla\psi_2)] \cdot \nabla(u_\eta - \psi_2)^+ \Big| dx \leq C\eta \|g_2^-\|^2_{L^2(\Omega)}$$

$$\frac{1}{\eta}\|(u_\eta - \psi_2)^+\|^2_{L^2(\Omega)} \leq C\eta\|g_2^-\|^2_{L^2(\Omega)}. \tag{27}$$

Proof. The proof is similar to the one of Lemma 3.6 by using $v = (u_\eta - \psi_2)^+$ in (11). □

We have $\quad \tilde{A}(u_\eta) - A(\psi_2) + a_0(u_\eta) - a_0(\psi_2) + z_\eta - \dfrac{1}{\eta}(u_\eta - \psi_1)^- = -g_2^+$

where $z_\eta = \dfrac{1}{\eta}(u_\eta - \psi_2)^+ - g_2^-$.

Our aim is to prove the strong convergence of z_η^+ to 0 in $L^2(\Omega)$.

Using z_ϵ^+ as test function in (11), one has

$$\int_\Omega \big[\tilde{a}(x,u_\eta,\nabla u_\eta) - \tilde{a}(x,\psi_2,\nabla\psi_2)\big]\nabla z_\eta^+ \mathrm{d}x + \int_\Omega \big[a_0(x,u_\eta) - a_0(x,\psi_2)\big] z_\eta^+ \mathrm{d}x$$
$$+ \int_\Omega |z_\epsilon^+|^2 \mathrm{d}x - \frac{1}{\eta}\int_\Omega (u_\eta - \psi_1)^- z_\eta^+ \mathrm{d}x = \langle -g_2^+, z_\epsilon^+\rangle \leq 0.$$

On the one hand, since $\psi_1 \leq \psi_2$ and $u_\eta > \psi_2$ on $\{z_\eta > 0\}$, one has $(u_\eta - \psi_1)^- z_\eta^+ = 0$ a.e. in Ω. Therefore

$$-\frac{1}{\eta}\int_\Omega (u_\eta - \psi_1)^- z_\eta^+ \mathrm{d}x = 0.$$

Since $u_\eta > \psi_2$ on $\{z_\eta > 0\}$ and a_0 is non decreasing with respect to the last argument, one has $\big[a_0(x,u_\eta) - a_0(x,\psi_1)\big] z_\eta^+ \geq 0$ a.e. in Ω. Therefore

$$\int_\Omega \big[a_0(x,u_\eta) - a_0(x,\psi_2)\big] z_\eta^+ \mathrm{d}x \geq 0.$$

On the other hand, note that, $\mathcal{B}$ being the set $\{\frac{1}{\eta}[(u_\eta - \psi_2)^+ - g_2^-] > 0\}$,

$$\begin{aligned}&\int_\Omega \left[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\right]\cdot\nabla z_\eta^+\mathrm{d}x\\&\quad=\int_\Omega 1_{\mathcal{B}}\left[\tilde{a}(x,u_\eta,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\right]\nabla\left[\frac{1}{\eta}[(u_\eta-\psi_2)^+-g_2^-]\right]\mathrm{d}x\\&\quad=\int_\Omega 1_{\mathcal{B}}\left[\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\right]\nabla\left[\frac{1}{\eta}[(u_\eta-\psi_2)^+-g_2^-]\right]\mathrm{d}x,\end{aligned}$$

since in this situation the integration holds in the set $\{u_\eta>\psi_2\}$. Thus,

$$\begin{aligned}&\Big[\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big]\nabla\left[\frac{1}{\eta}[(u_\eta-\psi_2)^+-g_2^-]\right]\\&\quad\geq\frac{1}{\eta}\Big[\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big]\nabla(u_\eta-\psi_2)\\&\qquad-\Big|\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big||\nabla g_2^-|\\&\quad\geq-\Big|\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big||\nabla g_2^-|.\end{aligned}$$

Thanks to (27), one gets

$$\Big[\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big]\nabla(u_\eta-\psi_2)^+\to 0 \text{ in } L^1(\Omega).$$

Similar arguments detailed previously yield,

$$\int_\Omega 1_{\{\frac{1}{\eta}[(u_\eta-\psi_2)^+-g_2^-]>0\}}\Big|\tilde{a}(x,\psi_2,\nabla u_\eta)-\tilde{a}(x,\psi_2,\nabla\psi_2)\Big||\nabla g_2^-|\mathrm{d}x\to 0.$$

As a conclusion, z_η^+ converges strongly to 0 in $L^2(\Omega)$.

Since $z_\eta=\mu_\eta^2-g_2^-$ and $\mu_\eta^2\geq 0$, this implies that

$$0\leq\mu_\eta^2\leq g_2^-+z_\eta^+,$$

since $z_\eta^+\to 0$ in $L^2(\Omega)$, μ_η^2 is bounded in $L^2(\Omega)$, by extracting a subsequence, there exists a non-negative function μ_2 such that

$$\mu_\eta^2\rightharpoonup\mu^2\quad in\, L^2(\Omega)\quad\text{resp. in}\,\mathcal{D}'(\Omega)\,\text{and}\quad 0\geq-\mu^2\geq-g_2^-.$$

By (26), one deduces

$$\mu^1-\mu^2=-div[a(x,u,\nabla u)]+a_0(x,u)-f.$$

We know already that $B(u)\in V_{p'(\cdot)}^*$ and we can add that

$$-(B(u))^-\geq-g_2^-=-(B(\psi_2))^-,$$

which implies that $B(u)\geq-(B(\psi_2))^-$. This completes the proof of full Lewy–Stampacchia's inequality (8) in the regular case.

Remark 3.10. As in Remark 3.8, one can prove Lewy–Stampacchia inequality without proving $B(u)\in V_{p'(\cdot)}^*$, following the same type of arguments.

3.3. Proof of Lewy–Stampacchia Inequalities in the General Case

Let us consider now general data as assumed in H_4. Thanks to [13, Section 3.3], there exist g_n^1 and g_n^2 such that:

$$\begin{cases} g_n^1 \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad g_n^1 \geq 0 \quad g_n^1 \to g_1^+ \quad \text{strongly in} \quad W^{-1,p'(\cdot)}(\Omega), \\ g_n^2 \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad g_n^2 \geq 0 \quad g_n^2 \to g_2^- \quad \text{strongly in} \quad W^{-1,p'(\cdot)}(\Omega). \end{cases} \tag{28}$$

3.3.1. The First Lewy–Stampacchia Inequality. Associated with g_n^1, denote the following f_n^1 by,

$$f_n^1 = A(\psi_1) + a_0(\psi_1) - g_n^1 + g_1^-, \quad g_1^- \in W^{-1,p'(\cdot)}(\Omega), \quad g_1^- \geq 0. \tag{29}$$

Note that $f_n^1 \in W^{-1,p'(\cdot)}(\Omega)$ and f_n^1 converges strongly to f in $W^{-1,p'(\cdot)}(\Omega)$. We also define B^n by

$$\forall v \in W^{1,p(\cdot)}(\Omega), \quad B^n(v) = A(v) + a_0(v) - f_n^1.$$

Then, $B^n(\psi_1) = g_n^1 - g_1^-$. By Proposition 3.5, there exists u_n in $K(\psi_1, \psi_2)$ such that for all v in $K(\psi_1, \psi_2)$, one has

$$\int_\Omega \Big[a(x, u_n, \nabla u_n)\nabla(v - u_n) + a_0(x, u_n)(v - u_n)\Big]\mathrm{d}x \geq \langle f_n^1, v - u_n\rangle. \tag{30}$$

Satisfying (see Sect. 3.2.1)

$$B(u_n) \in V^*_{p'(\cdot)}, \quad B(u_n) \leq (B(u_n))^+ \leq g_n^1. \tag{31}$$

Since this solution comes from the above penalization method, and C in (15) can be chosen independent of n, one gets that

$$\|u_n\|_{W_0^{1,p(\cdot)}(\Omega)} + \|a(x, u_n, \nabla u_n)\|_{(L^{p'(\cdot)}(\Omega))^d} \leq C.$$

Up to a subsequence denoted similarly,

$$\begin{cases} u_n \rightharpoonup u \quad in\, W_0^{1,p(\cdot)}(\Omega), \quad \text{strongly in}\, L^{p(\cdot)}(\Omega) \quad \text{and a.e. in}\, \Omega, \\ a_0(x, u_n) \to a_0(x, u) \quad \text{strongly in}\, L^{p'(\cdot)}(\Omega), \\ a(x, u_n, \nabla u_n) \rightharpoonup \chi \quad \text{weakly in}\, (L^{p'(\cdot)}(\Omega))^d. \end{cases}$$

Since $K(\psi_1, \psi_2)$ is a closed convex subset of $W_0^{1,p(\cdot)}(\Omega)$, one gets $u \in K(\psi_1, \psi_2)$.

Taking $v = u$ in (30), one has

$$\int_\Omega \Big[a(x, u_n, \nabla u_n)\nabla(u - u_n) + a_0(x, u_n)(u - u_n)\Big]\mathrm{d}x \geq \langle f_n^1, u - u_n\rangle, \tag{32}$$

and passing to the limit, we get

$$\limsup_n \int_\Omega a(x, u_n, \nabla u_n)\nabla u_n \mathrm{d}x \leq \int_\Omega \chi \nabla u \mathrm{d}x.$$

The pseudomonoticity of the operator $A(v) = -div[a(x, v, \nabla v)]$ yields

$\mathrm{div}\chi = div[a(x,u,\nabla u)]$ and

$$\lim_n \int_\Omega a(x,u_n,\nabla u_n)\nabla u_n \mathrm{d}x = \int_\Omega a(x,u,\nabla u)\nabla u \mathrm{d}x.$$

Arguments already detailed previously yield $\chi = a(x,u,\nabla u)$.

Passing to the limit in (30), there exists $u \in K(\psi_1,\psi_2)$ such that

$$\int_\Omega \Big[a(x,u,\nabla u)\nabla(v-u) + a_0(x,u)(v-u)\Big]\mathrm{d}x \geq \langle f, v-u\rangle,\ \forall v \in K(\psi_1,\psi_2).$$

Passing to the limit in $B(u_n) \leq g_n^1$, one gets $B(u) \leq g_1^+$ in $W^{-1,p'(\cdot)}(\Omega)$. Therefore,

$$\exists \kappa \in W^{-1,p'(\cdot)}(\Omega),\ \kappa = g_1^+ - B(u) \geq 0 \quad \text{such that}\, B(u) = g_1^+ - \kappa,$$

which implies $B(u) \in V^*_{p'(\cdot)}$. Since $(B(u_n))^+ \leq g_n^1$, one has at the limit $B(u)^+ \leq g_1^+$. Therefore, $B(u) \in V^*_{p'(\cdot)}$ and $B(u) \leq (B(u))^+ \leq g_1^+$.

This completes the proof of the first Lewy–Stampacchia inequality (6) of the main theorem.

3.3.2. The Second Lewy Stampacchia Inequality. Associated with g_n^2, denote the following f_n^2 by

$$f_n^2 = A(\psi_2) + a_0(\psi_2) - g_2^+ + g_n^2, \quad g_2^+ \in W^{-1,p'(\cdot)}(\Omega), \quad g_2^+ \geq 0. \tag{33}$$

Note that $f_n^2 \in W^{-1,p'(\cdot)}(\Omega)$ and f_n^2 converges strongly to f in $W^{-1,p'(\cdot)}(\Omega)$. We also define B^n by

$$\forall v \in W^{1,p(\cdot)}(\Omega), \quad B^n(v) = A(v) + a_0(v) - f_n^2.$$

Then, $B^n(\psi_2) = g_2^+ - g_n^2$. By proposition 3.5, there exists $u_n \in K(\psi_1,\psi_2)$, such that $\forall v \in K(\psi_1,\psi_2)$

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla(v-u_n)\mathrm{d}x + \int_\Omega a_0(x,u_n)(v-u_n)\mathrm{d}x \geq \langle f_n^1, v-u_n\rangle.$$

Satisfying (see Sect. 3.2.2),

$$B(u_n) \in V^*_{p'(\cdot)}, \quad B(u_n) \geq -(B(u_n))^- \geq -g_n^2. \tag{34}$$

By a similar proof to the one of Sect. 3.3.1, one gets that there exists $u \in K(\psi_1,\psi_2)$ such that

$$\int_\Omega a(x,u,\nabla u)\nabla(v-u)\mathrm{d}x + \int_\Omega a_0(x,u)(v-u)\mathrm{d}x \geq \langle f, v-u\rangle, \forall v \in K(\psi_1,\psi_2).$$

We know already that $B(u_n) \in V^*_{p'(\cdot)}$. Following Sect. 3.3.1, passing to the limit in $B(u_n) \geq -g_n^2$, one gets $B(u) \geq -g_2^-$ in $W^{-1,p'(\cdot)}(\Omega)$ and we can add that

$$B(u) \in V^*_{p'(\cdot)}, \quad -g_2^- \leq -(B(u))^- \leq B(u).$$

Remark 3.11. • By avoiding assumptions (2.7)–(2.9) of [13], this result is a significant generalization of Lewy–Stampacchia inequality for pseudomonotone operators.

- Note that in general the solution to the variational inequality is not a priori unique. So that, satisfying both Lewy–Stampacchia inequalities simultaneously is still an issue.

4. An Example of Problem Satisfying Both Lewy–Stampacchia's Inequalities

The aim of this section is to prove the last statement of the main theorem. For that, following [12], we propose a situation where both Lewy–Stampacchia inequalities are satisfied. Let j be a nonlinear superposition operator associated with a Carathéodory function denoted with the same name on $\Omega \times \mathbb{R}$ satisfying H_2 like a_0. One assumes moreover that it is strictly monotone ($\lambda \mapsto j(\cdot,\lambda)$ increasing). Let $U \in K(\psi_1,\psi_2)$ and note that $A(\psi_i)+a_0(\psi_i)+j(\psi_i)-f-j(U) = g_i^+ - g_i^- + j(\psi_i) - j(U) \in V^*_{p'(\cdot)}$ ($i=1,2$).

Then, from Sect. 3, there exist u_1 and u_2 in $K(\psi_1,\psi_2)$ satisfying, for any $v \in K(\psi_1,\psi_2)$,

$$\begin{cases} \displaystyle\int_\Omega a(\cdot,u_1,\nabla u_1)\nabla(v-u_1)+[a_0(u_1)+j(u_1)](v-u_1)\mathrm{d}x \\ \qquad \displaystyle\geq \langle f, v-u_1\rangle + \int_\Omega j(U)(v-u_1)\mathrm{d}x, \\ \displaystyle\int_\Omega a(\cdot,u_2,\nabla u_2)\nabla(v-u_2)+[a_0(u_2)+j(u_2)](v-u_2)\mathrm{d}x \\ \qquad \displaystyle\geq \langle f, v-u_2\rangle + \int_\Omega j(U)(v-u_2)\mathrm{d}x, \end{cases} \tag{35}$$

with the additional information that $B_j(u_i)-j(U)\in V^*_{p'(\cdot)}$ ($i=1,2$) where one denotes $B_j(u)=B(u)+j(u)$ and

$$\begin{cases} B_j(u_1)-j(U) \leq (B_j(u_1)-j(U))^+ \leq (B_j(\psi_1)-j(U))^+, \\ -(B_j(\psi_2)-j(U))^- \leq -(B_j(u_2)-j(U))^- \leq B_j(u_2)-j(U). \end{cases} \tag{36}$$

Assuming furthermore that the solution to (35) is unique (this can be obtained by adapting, e.g., the proof of [12, Prop. 2.2] in the framework of variable exponent Sobolev spaces), one gets that $u_1=u_2$. If moreover $U=u$ is chosen from the solutions given by Proposition 3.5, it is also a solution to (35) and $u=u_1=u_2$. Consequently, $B(u)\in V^*_{p'(\cdot)}$ and

$$-(B(\psi_2)+j(\psi_2)-j(u))^- \leq B(u) \leq (B(\psi_1)+j(\psi_1)-j(U))^+.$$

Since $\lambda\mapsto j(\cdot,\lambda)$ is an increasing function, $\psi_1\leq u\leq\psi_2$ yields

$$B(\psi_1)+j(\psi_1)-j(u)\leq g_1^+, \quad B(\psi_2)+j(\psi_2)-j(u)\geq -g_2^-$$

and $B(u)\in V^*_{p'(\cdot)}$ with

$$-(A(\psi_2)+a_0(\psi_2)-f)^- \leq A(u)+a_0(u)-f \leq (A(\psi_1)+a_0(\psi_1)-f)^+. \tag{37}$$

To finish, let us give examples of situations leading to the uniqueness of the solution. Consider u_1 and u_2, two given solutions to (35), and $p_\delta : \mathbb{R} \to \mathbb{R}$, a Lipschitz-continuous, non-decreasing function, such that $p_\delta(0) = 0$.

Denote by $w_1 = u_1 - \frac{1}{c} p_\delta(u_1 - u_2)$ where $c = \|p'_\delta\|_\infty$.

If $u_1 \geq u_2$, then $0 \leq \frac{1}{c} p_\delta(u_1 - u_2) \leq u_1 - u_2$ and $u_2 \leq w_1 \leq u_1$; similarly, if $u_1 \leq u_2$, then $u_1 \leq w_1 \leq u_2$.

Thanks to the chain rule, $w_1 \in K(\psi_1, \psi_2)$, as well as $w_2 = u_2 + \frac{1}{c} p_\delta(u_1 - u_2)$. Then, using w_1 in the first part of (35), w_2 in the second part of (35), and adding the corresponding inequalities, one gets that

$$\int_\Omega \Big[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)\Big] \nabla p_\delta(u_1 - u_2) \\ + [a_0(u_1) - a(u_2) + j(u_1) - j(u_2)] p_\delta(u_1 - u_2) \mathrm{d}x \leq 0.$$

Set $p_\delta(r) = \min \Big[1, \ln\left(\frac{re}{\delta}\right)^+\Big]$ (see, e.g., [21]). Note that p_δ is compatible with the assumptions, $p'_\delta(r) = \dfrac{1}{r} \mathbf{1}_{\left\{\frac{\delta}{e} < r < \delta\right\}}$ and it converges pointwise to the sign$^+$ function. So, Fatou's lemma yields

$$\liminf_\delta \int_\Omega [a_0(u_1) - a(u_2) + j(u_1) - j(u_2)] p_\delta(u_1 - u_2) \mathrm{d}x \geq \int_\Omega [j(u_1) - j(u_2)]^+ \mathrm{d}x.$$

Concerning the main operator, assume in a first case that a is Lipschitz continuous in the following sense: if $u \in W^{1,p(\cdot)}(\Omega)$,

$$|a(\cdot, t, \nabla u) - a(\cdot, s, \nabla u)| \leq \beta_1(\nabla u)|t - s| \quad \text{where } \beta_1(\nabla u) \in L^{p'(\cdot)}(\Omega).$$

$$\begin{aligned}
\text{Thus,} \quad & \int_\Omega \Big[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_2, \nabla u_2)\Big] \nabla p_\delta(u_1 - u_2) \mathrm{d}x \\
&= \int_\Omega p'_\delta(u_1 - u_2) \Big[a(\cdot, u_1, \nabla u_1) - a(\cdot, u_1, \nabla u_2)\Big] \nabla(u_1 - u_2) \mathrm{d}x \\
&\quad + \int_\Omega p'_\delta(u_1 - u_2) \Big[a(\cdot, u_1, \nabla u_2) - a(\cdot, u_2, \nabla u_2)\Big] \nabla(u_1 - u_2) \mathrm{d}x \\
&\geq \int_\Omega p'_\delta(u_1 - u_2) \Big[a(\cdot, u_1, \nabla u_2) - a(\cdot, u_2, \nabla u_2)\Big] \nabla(u_1 - u_2) \mathrm{d}x \\
&\geq -\int_\Omega p'_\delta(u_1 - u_2) |u_1 - u_2| \beta_1(\nabla u_2) |\nabla(u_1 - u_2)| \mathrm{d}x \\
&\geq -\int_{\frac{\delta}{e} < u_1 - u_2 < \delta} \beta_1(\nabla u_2) |\nabla(u_1 - u_2)| \mathrm{d}x.
\end{aligned}$$

Assume in a second case a Hölder-continuous property with a stronger monotony in the following sense[1]:

$$\begin{aligned}
& [a(\cdot, \lambda, \vec{\xi}_1) - a(\cdot, \lambda, \vec{\xi}_2)](\vec{\xi}_1 - \vec{\xi}_2) \geq c_0 |\vec{\xi}_1 - \vec{\xi}_2|^{\alpha(\cdot)}, \\
& |a(\cdot, t, \nabla u) - a(\cdot, s, \nabla u)| \leq \beta_2(\nabla u)|t - s|^{\theta(\cdot)} \\
& \text{where } c_0 > 0, \quad \alpha' \geq \frac{1}{\theta} > 1 \text{ and } \beta_2(\nabla u) \in L^{\alpha'(\cdot)}(\Omega) \text{ if } u \in W^{1,p(\cdot)}(\Omega).
\end{aligned}$$

[1] Additional assumptions are made on the exponents to make sense to the integrals.

$$\begin{aligned}
\text{Thus, } &\int_\Omega \Big[a(\cdot,u_1,\nabla u_1)-a(\cdot,u_2,\nabla u_2)\Big]\nabla p_\delta(u_1-u_2)\mathrm{d}x\\
&\geq c_0\int_\Omega p_\delta'(u_1-u_2)|\nabla(u_1-u_2)|^{\alpha(\cdot)}\mathrm{d}x - c_0\int_\Omega p_\delta'(u_1-u_2)|\nabla(u_1-u_2)|^{\alpha(\cdot)}\mathrm{d}x\\
&\quad - C\int_\Omega p_\delta'(u_1-u_2)|u_1-u_2|^{\theta(\cdot)\alpha'(\cdot)}|\beta_2(\nabla u_2)|^{\alpha'(\cdot)}\mathrm{d}x\\
&\geq -C\int_{\frac{\delta}{e}<u_1-u_2<\delta}|u_1-u_2|^{\theta(\cdot)\alpha'(\cdot)-1}|\beta_2(\nabla u_2)|^{\alpha'(\cdot)}\mathrm{d}x\\
&\geq -C\int_{\frac{\delta}{e}<u_1-u_2<\delta}|\beta_2(\nabla u_2)|^{\alpha'(\cdot)}\mathrm{d}x,
\end{aligned}$$

where C is a positive constant independent of δ.

In both situations, Lebesgue theorem yields

$$\liminf_\delta \int_\Omega \Big[a(\cdot,u_1,\nabla u_1)-a(\cdot,u_2,\nabla u_2)\Big]\nabla p_\delta(u_1-u_2)\mathrm{d}x \geq 0$$

and $\int_\Omega [j(u_1)-j(u_2)]^+\mathrm{d}x = 0$.

As j is increasing with respect to its second argument, one gets that $u_1 \leq u_2$. Interchanging u_1 and u_2, the result of uniqueness holds.

We invite the reader interested in more general situations, like local continuity assumptions, to consult [12] concerning the bilateral problem and [5,14] and their references for uniqueness methods.

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