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ENTROPY SOLUTIONS FOR FIRST-ORDER QUASILINEAR EQUATIONS RELATED TO A BILATERAL OBSTACLE CONDITION IN A BOUNDED DOMAIN

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Abstract

This paper is devoted to the existence and the uniqueness of the entropy solution for a general scalar conservation law associated with a forced bilateral obstacle condition in a bounded domain of $\mathbb{R}^p$, $p \geq 1$.

The method of penalization is used with a view to obtaining an existence result. However, the former only gives uniform L^∞-estimates and so leads in fact to look for an Entropy Measure-Valued Solution, according to the specific properties of bounded sequences in L^∞. The uniqueness of this EMVS is proved. Classically, it first ensures the existence of a bounded and measurable function U entropy solution and then the strong convergence in L^q of approximate solutions to U.

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§1. Introduction

1.1. Physical Motivations

A.Bensoussan & J.L.Lions[3] have first introduced obstacle problems for first-order hyperbolic operators, as part of the study of cost-functions associated with deterministic processes. Since then, numerous researches have been carried out on this matter. Among such works, we can quote those of J.I.Diaz & L.Veron[5], using the properties of nonlinear semigroups of contractions in L^1, or L.Barthelemy's [2] referring to nonlinear subpotential operators. In [11], F.Mignot & J.P.Puel have developed the method of penalization for linear variational and quasi-variational inequalities. This very technique is used in [8] for the Dirichlet problem when it comes to a nonlinear operator related to a unilateral constraint.

This paper deals with the non homogeneous Dirichlet problem for a general scalar conservation law associated with a bilateral forced constraint. The physical motivations of such a study are diverse as soon as one is interested in the evolution of a heterogeneous phase

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through porous media. For example, in the hydrological field, within the context of the research on liquid transfers through the soil, we are interested in the evolution of any effluent c within the flow of various substances moving in the subsoil. The first simplified modelling consists in taking into account but one phase saturating the soil, made of two components without any chemical interactions : water and the component c. Given that the distribution of temperatures T and the pressure field P of the fluid phase are determined, sufficiently smooth functions, the transcription of c's mass conservation law provides the equality ruling the component c's mass fraction ω_c :

$$\partial_t \omega_c - \frac{k(x)}{\mu(\omega_c)} \vec{\nabla} \omega_c . \left(\vec{\nabla} P - \rho\left(T, \omega_c\right) \vec{g} \right) = 0. \tag{1.1}$$

In the latter equation, $k(x)$ denotes the absolute permeability at the point x, μ being the dynamic viscosity of the fluid phase and $\rho\left(T, \omega_c\right)$ its voluminal mass, defined by the composition ω_c at the temperature T. Lastly, $\vec{g}$ is the gravity acceleration vector. Moreover, depending on the geological nature of the subsoil, the molecular diffusion-dispersion effects have been neglected in favor of the transport of effluent ones. Furthermore, ω_c must satisfy the bilateral obstacle condition :

$$\theta_{1,c}\left(T\left(t,x\right), P\left(t,x\right)\right) \leq \omega_c\left(t,x\right) \leq \theta_{2,c}\left(T(t,x), P\left(t,x\right)\right), \tag{1.2}$$

where $\theta_{1,c}\left(T\left(t,x\right), P\left(t,x\right)\right)$ and $\theta_{2,c}\left(T\left(t,x\right), P\left(t,x\right)\right)$ are two extreme saturation points at the temperature T and for the presure P. Indeed, beyond these values, the appearance of a new phase (liquid or solid) for the same number of components changes the thermodynamical nature of the considered system and this latter can not be described through the simplified continuity equation (1.1).

1.2. Mathematical Formulation

Regarding the first stage in the study of the equation (1.1) related to the bilateral constraint (1.2) and to a Dirichlet boundary condition, a change of variables by translation reveals a new unknown as well as a first-order hyperbolic operator including a reaction term depending on the lower obstacle. So the next operator is considered :

$$\mathbb{H}\left(t,x,.\right) : u \rightarrow \partial_t u + \sum_{i=1}^{p} \partial_{x_i}\left(f\left(u\right) B_i\left(t,x\right)\right) + g\left(t,x,u\right),$$

where only the dependence on u is taken into consideration in the transport and reaction terms.

Then, the model problem is the following : given that u_0, u^B and θ such that $0 \leq u_0 \leq \theta\left(0,.\right)$ a.e. on Ω, $0 \leq u^B \leq \theta$ a.e. on $]0,T[\times \partial\Omega$, where Ω is a subdomain of $\mathbb{R}^p$, $1 \leq p$ and $0 < T < +\infty$, find u satisfying the formal free boundary problem $(\mathcal{P})$:

$$0 \leq u \leq \theta \text{ in } Q =]0,T[\times \Omega, \tag{1.3}$$

$$\mathbb{H}\left(t,x,u\right) = 0 \text{ in } \left\{(t,x) \in Q,\ 0 < u\left(t,x\right) < \theta\left(t,x\right)\right\}, \tag{1.4}$$

$$u(t,\sigma) = u^B \text{ on a part of } \Sigma =]0,T[\times \partial\Omega,\ u(0,.) = u_0 \text{ in } \Omega.$$

The method of penalization is applied with a view to obtaining an existence result for the problem $(\mathcal{P})$. Hence, for each value of the parameter $\eta > 0$ (intended to tend to zero), one considers the weak entropy solution of the penalized problem $(\mathcal{P})_\eta$, resulting from the works of C.Bardos, A.Y.LeRoux & J.C.Nedelec[1] : find a bounded function u_η with bounded variations on Q such that

$$\mathbb{H}(t,x,u_\eta) + \frac{1}{\eta}\beta(t,x,u_\eta) = 0 \text{ in } Q, \tag{1.5}$$
$$u_\eta = u^B \text{ on a part of } \Sigma,\ u_\eta(0,\cdot) = u_0 \text{ in } \Omega,$$

where $\beta(t,x,u) = -u^- + (u - \theta(t,x))^+$. The equality (1.5) characterizes the penalization of (1.4) and the free boundary problem $(\mathcal{P})$ is regularized by adding a term which becomes dominating when η goes to 0^+.

We look for estimates of u_η which are independent from η. The well-known irregularity of the solutions to nonlinear first-order hyperbolic problems let one think that such estimates must be searched in the space $\mathrm{BV}(Q)\cap L^\infty(Q)$ of bounded functions with bounded variations in Q. But since the penalization operator $\beta(\cdot,\cdot,\cdot)$ depends on time and space variables, the classical methods used (see e.g. [7] or [8]) even permit to estimate the uniform bounds of u_η. So, to pass to the η-limit in the nonlinar terms of $\mathbb{H}(\cdot,\cdot,\cdot)$ we must precise some specific properties of bounded sequences in L^∞.

1.3. Some Reminders on Bounded Sequences in L^∞

Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^{p+1}$ and let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^\infty(\mathcal{O})$. Since the works of L.Tartar[13], it has been possible to specify the behavior of the sequence $(f(u_n))_{n\in\mathbb{N}}$, for all continuous functions f on $\mathbb{R}$. The proof is based on the properties of weak-$*$ topology on the space of Radon measures. Moreover, it uses the "disintegration" of a Young measure with respect to the Lebesgue measure on $\mathbb{R}^{p+1}$. All those results lead to the next compacity result :

Property 1.1. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of $L^\infty(\mathcal{O})$ such that*

$$\exists M > 0, \forall n \in \mathbb{N}, \|u_n\|_\infty \le M.$$

Then, there exists a subsequence, still denoted by $(u_n)_{n\in\mathbb{N}}$, extracted from $(u_n)_{n\in\mathbb{N}}$ and $(\nu_w)_{w\in\mathcal{O}}$ a family of probability measures on $\mathbb{R}$ with a support in $[-M,M]$, such that, for all bounded Caratheodory functions ψ on $\mathcal{O}\times[-M,M]$, the sequence $(\psi(\cdot,u_n(\cdot)))_{n\in\mathbb{N}}$ converges in $L^\infty(\mathcal{O})$ weak-$*$ towards the element :

$$w \to \int_{\mathbb{R}} \psi(x,\lambda)\, d\nu_w(\lambda).$$

The map $\nu : w \to \nu_w$ is called "Young measure associated with the sequence $(u_n)_{n\in\mathbb{N}}$" and its "disintegration" with respect to the Lebesgue measure on $\mathbb{R}$ is given by the relation

$$d\nu(\lambda,w) = d\nu_w(\lambda)\, dw.$$

Such a result has found its first application in [13] within the context of the approximation of a first-order quasilinear equation through an artificial viscosity method. In [4], R.J.Diperna studies the propagation by a scalar conservation law of a Dirac initial datum and introduces the notion of "admissible" (or entropy) measure-valued solution. In the sense of this formulation, the Young measure ν related to the sequence of approximate solutions $(u_\epsilon)_\epsilon$ is an admissible measure-valued solution which is reduced to a Dirac mass at $t = 0$.

So, by demonstrating that ν is also a Dirac mass solution, some general considerations prove that the bounded sequence $(u_\epsilon)_\epsilon$ strongly converges in L^q, $1 \leq q < +\infty$, to the point of concentration of ν. Namely, it establishes the existence of a bounded and measurable function u such that $d\nu_w = \delta_{u(w)}$ a.e. w in $\mathcal{O}$, where δ_z is the Dirac mass centered on the point z.

This reasonning has especially been applied to the numerical analysis of transport equations. Indeed, it is well-known that most numerical schemes (e.g. Finite Volume Scheme) only give an L^∞-estimate uniformly with respect to the mesh length Δx of the approximate solution $u_{\Delta x}$. In order to take the limit when Δx goes to 0^+, we are led to introduce the notion of measure-valued solution (or entropy process solution[6]).

Initially used in the case of the Cauchy problem, those methods can be adapted to the situation of a non homogeneous Dirichlet boundary condition, since the notion of measure-valued solution has been extended to bounded domains of $\mathbb{R}^p$ by A.Szepessy[12] and G.Vallet[14].

1.4. Entropy Measure-Valued Solution

On the one hand, in the case of a general first-order quasilinear equation, it is classical to introduce the notion of an entropy solution. This criterium - which warrants the uniqueness - selects, through all the weak solutions (i.e. in sense of distributions on Q) the most physically acceptable one as soon as some discontinuities appear. Moreover, it is a fact that if the initial datum is bounded, so it is with the solution to this kind of equation, under some assumptions on the source term; but the introduction of a constraint on the initial datum does not a priori pass on to the solution (some behavior and stability properties with respect to the associated control in the case of the Cauchy problem on $\mathbb{R}^p$ can be found in [9]). That is why we need an entropic formulation allowing for this constraint.

On the other hand, one of the major difficulties linked to the notion of measure-valued solution is to define its behavior on the boundary of the studied field. Given this definition, we must reformulate the boundary conditions by copying the one that has been provided by F.Otto in [10], for the most regular case in which a bounded solution u can be found. For this purpose, we will take advantage of the fact that if an entropy inequality is satisfied inside an open subset, then it is possible to define some integrals on its boundary, through a judicious change of variables.

Firstly, let us describe the behavior of a Young measure associated with the sequence $(u_\eta)_{\eta>0}$ inside the open subset Q. To clarify the writing, we note in the rest of this paper :

$$\begin{aligned}
\vec{F}(t,x,u,k) &= \operatorname{sign}(u-k)\,[f(u)-f(k)]\,\vec{B}(t,x)\,,\\
G(t,x,u,v) &= \operatorname{sign}(u-v)\left[\vec{\nabla}\cdot\left(f(v)\,\vec{B}(t,x)\right)+g(t,x,u)+\partial_t v\right],\\
\mathcal{L}(t,x,u,k,v) &= |u-k|\,\partial_t v+\vec{F}(t,x,u,k)\cdot\vec{\nabla}v-G(t,x,u,k)\,v.
\end{aligned}$$

Thus, according to the definitions given in [9] for the Cauchy problem, we say

Definition 1.1. *A Young measure ν is called an entropy measure-valued solution* (*emvs*) *to* (1.3) *and* (1.1) *if and only if*

$$\operatorname{Supp}\left(d\nu_{(t,x)}\right) \subset [0,\theta(t,x)] \quad \text{for a.e.}\ (t,x)\ \textit{in}\ Q, \tag{1.6}$$

and for all positive functions ξ *of* $H_0^1(Q)$, *for any real number* k *of* $[0,1]$,

$$\int_{Q\times\mathbb{R}} \mathcal{L}\left(t,x,\lambda,k\theta\left(t,x\right),\xi\right)\, d\nu_{(t,x)}\left(\lambda\right) dxdt \geq 0. \tag{1.7}$$

In order to formulate the initial condition in $L^1(\Omega)$ within the context of a measure-valued solution, one may notice that

Property 1.2 (Initial Condition). *Let* ν *be an emvs to* (1.3) *and* (1.4). *Then, for all positive functions* ζ *of* $L^1(\Omega)$ *and all functions* w *of* $L^\infty(\Omega)$,

$$\lim_{t\to 0^+} \operatorname{ess} \int_{\Omega\times\mathbb{R}} \left|\lambda - w\left(x\right)\theta\left(t,x\right)\right| \zeta\, d\nu_{(t,x)}\left(\lambda\right) dx \ \textit{ exists.}$$

Proof. As in [14], considering in the entropy relation (1.7) the test-function $\xi(t,x) = \varphi(t)\zeta(x)$, where φ and ζ belong respectively to $\mathcal{D}_+(0,T)$ and $\mathcal{D}_+(\Omega)$. The inequality obtained in this way proves that the bounded function

$$h^{k,\zeta} : t \to \int_{\Omega\times\mathbb{R}} \left|\lambda - k\theta\right| \zeta\left(x\right)\, d\nu_{(t,x)}\left(\lambda\right) dx$$

has a bounded variation on $[0,T]$. So, it is meaningful to consider its essential limit when t goes to 0^+. Moreover, $\zeta \to h^{k,\zeta}(t)$ is a continuous linear map, uniformly with respect to t. Then, a density argument leads to the existence of this essential limit for all positive functions ζ of $L^1(\Omega)$. Now, let $w_m = \sum_{i=0}^{m} k_i \mathbb{I}_{\mathcal{B}_i}$ be a simple function on Ω where k_i belongs to $\mathbb{Q}$ and $(\mathcal{B}_i)_{i\in\{1,\cdots,m\}}$ is a Borelian partition of Ω. Then,

$$\int_{\Omega\times\mathbb{R}} \left|\lambda - w_m\theta\right| \zeta\, d\nu_{(t,x)}\left(\lambda\right) dx = \sum_{i=1}^{m} \int_{\Omega\times\mathbb{R}} \left|\lambda - k_i\theta\right| \mathbb{I}_{\mathcal{B}_i} \zeta d\nu_{(t,x)}\left(\lambda\right) dx.$$

Consequently, the essential limit of $h^{k,\zeta}(t)$, when t goes to 0^+, exists for all positive functions ζ of $L^1(\Omega)$ and for all simple functions w on Ω with values in $\mathbb{Q}$. Lastly, it can be extended to all functions w of $L^\infty(\Omega)$, since w can be considered as an $L^\infty(\Omega)$-limit of such a simple function's sequence and since, for all w and $\hat{w}$ of $L^\infty(\Omega)$, $\left|h^{w,\zeta}(t) - h^{\hat{w},\zeta}(t)\right| \leq \|\zeta\theta\|_{L^1(\Omega)} \|w-\hat{w}\|_{L^\infty(\Omega)}$, independently from t in $[0,T]$.

Now, we have to give a sense to the trace of a Young measure ν on Γ. Let us come back to the idea introduced by A.Szepessy[12] and let us denote $\mathcal{H}^p$ is the p-dimensional Hausdorff measure on Σ.

Definition 1.2. *Let* ϵ *be a strictly positive real number and let* ν *be some Young measure with support in* $[-M,M]$. *We consider the change of coordinates* $x \to (\sigma,\tau)$ *for* x *in a neighborhood of* Γ *:*

$$x = \sigma - \tau\, \vec{n}\left(\sigma\right),$$

where (σ,τ) *belongs to* $\Gamma\times]0,\epsilon[$. *We denote by* $J(\sigma,\tau)$ *the Jacobian determinant associated with this change of coordinates.*

Then, there exist a sequence $(\tau_i)_{i\in\mathbb{N}}$ *in* $]0,\epsilon[$ *which tends to* 0^+ *and a Young measure* μ *on* Γ, *called a Young measure trace on* Σ *for* ν, *described by its disintegration form :*

$$d\mu\left(t,\sigma,\lambda\right) = d\mu^{\nu}_{(t,\sigma)}\left(\lambda\right) d\mathcal{H}^p,$$

where $d\mu^{\mu}_{(t,\sigma)}$ is a family of probabilities on $\mathbb{R}$ with support in $[-M,M]$, such that,

$$\lim_{i\to+\infty}\int_{\Sigma}\int_{\mathbb{R}}\psi\left(t,\sigma,\lambda\right)\,d\nu_{(t,x_{(\sigma,\tau_i)})}\left(\lambda\right)J\left(\sigma,\tau_i\right)d\mathcal{H}^p=\int_{\Sigma}\int_{\mathbb{R}}\psi\left(t,\sigma,\lambda\right)\,d\mu\left(t,\sigma,\lambda\right)$$

for all bounded Caratheodory function ψ on $\Sigma\times\mathbb{R}$.

Then, the behaviour of an emvs in the neighborhood of the boundary Γ can be described through the next statement:

Property 1.3 (boundary condition). *Let ν be an emvs to* (1.4) *and* (1.3) *and let μ be a Young measure trace related to ν. Let us denote $\bar{x}=x_{(\sigma,\tau)}$. Then, for all positive functions ζ of $L^1(\Sigma)$ and for all functions w of $L^\infty(\Sigma)$,*

$$\lim_{\tau\to0^+}\text{ess}\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\bar{x},\lambda,w\left(t,\sigma\right)\theta\left(t,\bar{x}\right)\right)\cdot\vec{n}\,\zeta\,d\nu_{(t,\bar{x})}\left(\lambda\right)J\left(\sigma,\tau\right)d\mathcal{H}^p$$
$$=\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\sigma,\lambda,w\theta\right)\cdot\vec{n}\,\zeta d\mu\left(t,\sigma,\lambda\right).$$

Proof. The idea is the same as that already developed in the proof of property 1.2: one considers in the entropy inequality (1.7) the test-function $\xi\left(t,x\right)=\varphi\left(\tau\right)\zeta\left(t,\sigma\right)$, where φ and ζ belong to $\mathcal{D}_+\left(0,\epsilon\right)$ and $\mathcal{D}_+\left(\Sigma\right)$. Then

$$\int_0^\epsilon\varphi\left(\tau\right)\int_{\Sigma\times\mathbb{R}}\{\left|\lambda-k\theta\left(t,\bar{x}\right)\right|\partial_t\zeta\left(t,\sigma\right)+\vec{F}\left(t,\bar{x},\lambda,k\theta\left(t,\bar{x}\right)\right)\cdot\vec{\nabla}\zeta\left(t,\sigma\right)$$
$$-\,\text{sign}\left(\lambda-k\theta\left(t,\bar{x}\right)\right)G\left(t,\bar{x},\lambda,k\theta\left(t,\bar{x}\right)\right)\zeta\left(t,\sigma\right)\}\,d\nu_{(t,\bar{x})}\left(\lambda\right)J\left(\sigma,\tau\right)d\mathcal{H}^pd\tau$$
$$\geq\int_0^\epsilon\varphi'\left(\tau\right)\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\bar{x},\lambda,k\theta\left(t,\bar{x}\right)\right)\cdot\vec{n}\,\zeta\left(t,\sigma\right)\,d\nu_{(t,\bar{x})}\left(\lambda\right)J\left(\sigma,\tau\right)d\mathcal{H}^pd\tau.$$

The same arguments as before lead to the existence of the essential limit, when τ goes to 0^+, of the function

$$h^{w,\zeta}\left(\tau\right)=\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\bar{x},\lambda,w\theta\left(t,\bar{x}\right)\right).\vec{n}\,\zeta\left(t,\sigma\right)\,d\nu_{(t,\bar{x})}\left(\lambda\right)J\left(\sigma,\tau\right)d\mathcal{H}^p$$

for every positive function ζ in $L^1(\Sigma)$ and for all simple function w on Σ with values in $\mathbb{Q}$. Moreover, since f is continuous, one has

$$\lim_{\tau\to0^+}\text{ess}\,h^{w,\zeta}\left(\tau\right)=\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\sigma,\lambda,w\left(t,\sigma\right)\theta\left(t,\sigma\right)\right).\vec{n}\,\zeta\left(t,\sigma\right)\,d\mu\left(t,\sigma,\lambda\right).$$

Let w be an element of $L^\infty(\Sigma)$ and let $(w_m)_{m\in\mathbb{N}}$ be a sequence of simple functions with values in $\mathbb{Q}$ that converges a.e. and in $L^\infty(\Sigma)$ to w. Then w and w_m are bounded a.e. by the same constant, independently from m. Let ϵ be a strictly positive real number. Since $(w_m)_{m\in\mathbb{N}}$ converges uniformly to w on Σ, for m large enough one has

$$\left|h^{w,\zeta}\left(\tau\right)-h^{w_m,\zeta}\left(\tau\right)\right|\leq C^{st}\epsilon\left\|\zeta\right\|_{L^1(\Sigma)},$$

where C^{st} is a constant independent from any parameter (m or τ). Thus, $h^{w_m,\zeta}(\tau)$ converges to $h^{w,\zeta}(\tau)$ as m goes to infinity, uniformly with respect to τ. So, the essential limit of $h^{w,\zeta}(\tau)$, when τ tends to 0^+, can be extended to all positive functions ζ of $L^1(\Sigma)$ and to any function w of $L^\infty(\Sigma)$.

Remark 1.1. A Young measure trace μ corresponding to ν is not unique but the value of the integral $\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\sigma,\lambda,w\right).\vec{n}\left(\sigma\right)\,d\mu\left(t,\sigma,\lambda\right)$ is the same for all Young measure traces μ associated with the emvs ν and for all w of $L^\infty(\Sigma)$ (see [12]).

Within the framework of the study of first-order quasilinear conservation laws, it is well-known that the boundary conditions can be included in the entropy inequality and are formulated by choosing a particular test-function. Here, we have chosen a mathematical formulation for $(\mathcal{P})$ by dissociating the behavior of a measure-valued solution ν inside the open subset Q (via the definition of an emvs) and on its boundary (by introducing a Young measure trace μ related to ν and by translating F.Otto's formulation of Dirichet boundary conditions[10] into the language of measure solutions). Hence we give the following definition.

Definition 1.3. *A Young measure ν is called an Entropy Measure-Valued Solution (EMVS) to $(\mathcal{P})$ if and only if it is an emvs to* (1.3) *and* (1.4) *and if the boundary conditions are fulfilled in the following sense : for all real numbers k of $[0,1]$, for all positive functions ζ of $L^1(\Sigma)$ and ς of $L^1(\Omega)$, for all Young measure traces μ on Σ related to ν,*

$$\begin{aligned}&-\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\sigma,\lambda,k\theta)\cdot\vec{n}\,\zeta\,d\mu(t,\sigma,\lambda)\\&\leq\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\sigma,\lambda,u^B\right)\cdot\vec{n}\,\zeta\,d\mu(t,\sigma,\lambda)-\int_{\Sigma}\vec{F}\left(t,\sigma,k\theta,u^B\right)\cdot\vec{n}\,\zeta\,d\mathcal{H}^p.\end{aligned}\tag{1.8}$$

$$\lim_{t\to0^+}\text{ess}\int_{\Omega}|\lambda-u_0(x)|\,\varsigma\,d\nu_{(t,x)}(\lambda)\,dx=0.\tag{1.9}$$

In paragraph 2, we are going to prove the existence of an EMVS ν to $(\mathcal{P})$, through the penalization method. Nonetheless, in order to demonstrate the uniqueness of ν and establish some sensitivity properties with respect to the initial and the boundary conditions, a global weak entropy formulation on $Q\cup\Sigma$ is necessary. Thus, let us note that the next equivalent definition holds.

Theorem 1.1. *A Young measure ν is an EMVS to $(\mathcal{P})$ if and only if it fulfills the constraint on the support* (1.6) *and if for all real numbers k of $[0,1]$, for all positive functions ξ of $H^1(Q)$, for all Young measure trace μ on Σ related to ν, the following relation holds*:

$$\begin{aligned}&\int_{Q\times\mathbb{R}}\mathcal{L}(t,x,\lambda,k\theta,\xi)\,d\nu_{(t,x)}(\lambda)\,dxdt+\int_{\Omega}|u_0-k\theta(0,\cdot)|\,\xi(0,\cdot)\,dx\\&\geq-\int_{\Sigma\times\mathbb{R}}\vec{F}\left(t,\sigma,\lambda,u^B\right)\cdot\vec{n}\,\xi\,d\mu(t,\sigma\lambda)+\int_{\Sigma}\vec{F}\left(t,\sigma,k\theta,u^B\right)\cdot\vec{n}\,\xi\,d\mathcal{H}^p.\end{aligned}\tag{1.10}$$

Proof. (i) Clearly, if ν is an EMVS to $(\mathcal{P})$, then it is an emvs to (1.3) and (1.4). Furthermore, in order to establish the boundary condition (1.8), let us consider φ an element of $\mathcal{D}_+(0,T)$, ζ a positive function of $L^1(\Sigma)$ and let ϕ_m be the Lipschitzian approximation of $\mathbb{I}_\Omega$ defined for an m large enough by $\phi_m(\bar{x})=\min(m\tau,1)$, with $\bar{x}=x_{(\sigma,\tau)}$ in the coordinates (σ,τ) introduced in the Definition 1.2. Thus, we choose in the entropy inequality (1.7) the test-function

$$\xi(t,\bar{x})=(1-\phi_m(\bar{x}))\,\varphi(t)\,\zeta(\sigma).$$

This leads to consider the next integral :

$$m\int_0^{1/m}\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\bar{x},\lambda,k\theta(t,\bar{x}))\cdot\vec{n}\zeta\,d\nu_{(t,\bar{x})}(\lambda)\,J(\sigma,\tau)\;d\tau d\mathcal{H}^p,$$

whose limit, when m tends to infinity, is obtained by referring to Property 1.3.

The initial condition (1.9) is treated in the same way as in R.Eymard, T.Gallouet & R.Herbin's [6] through the one of S.N.Kruskov[7]: let τ be an element of $]0,T[$, let φ_m be a regular approximation of $1-\mathbb{I}_{]0,\tau[}$. Let ζ be a function of $\mathcal{D}_+(\Omega)$ and $\rho_{p,l}$ be the standard mollifer sequence (see Section 3, § 3.1). We choose in the entropy inequality (1.10) the test-function

$$\xi(t,x)=\zeta(x)\,\rho_{p,l}(x-y)\,\varphi_m(t) \text{ and } k=k(y)=\begin{cases}\frac{u_0(y)}{\theta(0,y)}, & \text{if } \theta(0,y)>0,\\ 0, & \text{else.}\end{cases}$$

That way $u_0(y)=k(y)\,\theta(0,y)$ for a.e. y in Ω. The relation (1.9) is obtained by integrating with respect to the variable y and by taking successively the limit when m and l tend to infinity.

(ii) Conversely, let ζ be an element of $\mathcal{D}_+\left(\bar{Q}\right)$ and ϕ a positive function of $H_0^1(\Omega)\cap L^\infty(\Omega)$. For all m of $\mathbb{N}$, we introduce φ_m the Lipschitzian approximation of $\mathbb{I}_{]0,T[}$ given by $\varphi_m(t)=\max(0,\min(mt,m(T-t),1))$. We consider in the entropy inequality (1.7) the test-function $\xi(t,x)=\zeta(t,x)\,\phi(x)\,\varphi_m(t)$. Hence we have the next relation:

$$\int_{Q\times\mathbb{R}}\mathcal{L}(t,x,\lambda,k\theta,\zeta\phi)\,\varphi_m\,d\nu_{(t,x)}(\lambda)\,dxdt\geq -m\int_0^{1/m}\int_{\Omega\times\mathbb{R}}\zeta\phi\,|\lambda-k\theta|\,d\nu_{(t,x)}(\lambda)\,dxdt.$$

Let us denote by I_m the right-hand side integral in the previous inequality. Then, there exists a constant C, independent from m, such that

$$\left|I_m+m\int_0^{1/m}\int_\Omega\zeta(0,x)\,\phi\,|u_0-k\theta(0,x)|\,dxdt\right|$$
$$\leq Cm\int_0^{1/m}\int_\Omega\left\{|\zeta(0,\cdot)-\zeta(t,\cdot)|+|\theta(0,\cdot)-\theta(t,\cdot)|+\int_{\mathbb{R}}|\lambda-u_0|\,d\nu_{(t,x)}(\lambda)\right\}dxdt.$$

That way, according to the definition of the initial condition (1.9) and to the regularity of the functions ζ and $\theta$when m goes to infinity. Thus, for all functions ζ of $\mathcal{D}_+\left(\bar{Q}\right)$ and all ϕ elements of $H_0^1(\Omega)\cap L^\infty(\Omega)$,

$$\int_{Q\times\mathbb{R}}\mathcal{L}(t,x,\lambda,k\theta,\zeta\phi)\,d\nu_{(t,x)}(\lambda)\,dxdt\geq -\int_\Omega\zeta(0,x)\,|u_0-k\theta(0,x)|\,\phi dx.$$

Let us now assume that ϕ is the Lipschitzian approximation of $\mathbb{I}_\Omega$ defined for an m large enough by $\phi(\bar{x})=\min(m\tau,1)$, with $\bar{x}=x_{(\sigma,\tau)}$ in the coordinates (σ,τ) introduced in Definition 1.2. This leads to consider, as previously, the following term:

$$m\int_0^{1/m}\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\bar{x},\lambda,k\theta(t,\bar{x}))\cdot\vec{n}\zeta\,d\nu_{(t,\bar{x})}(\lambda)\,J(\sigma,\tau)\,d\tau d\mathcal{H}^p,$$

whose limit, when m tends to infinity, is obtained by referring to Property 1.3.
Finally, for any function ζ of $\mathcal{D}_+\left(\bar{Q}\right)$,

$$\int_{Q\times\mathbb{R}}\mathcal{L}(t,x,\lambda,k\theta,\zeta)\,d\nu_{(t,x)}(\lambda)\,dxdt$$
$$\geq\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\bar{x},\lambda,k\theta)\,.\vec{n}\zeta\,d\mu(t,\sigma,\lambda)-\int_\Omega\zeta(0,.)\,|u_0-k\theta(0,.)|\,dx.$$

By density, this inequality is still true for all positive functions ζ of $H^1(Q)$ and to complete the proof, we only need to refer to boundary conditions (1.8).

§2. Existence Result

We decide to adapt the concept of measure-valued solution to the situation where the approximating sequence $(u_\eta)_{\eta>0}$ is the sequence of weak entropy solutions of penalized problems $\left[(\mathcal{P})_\eta\right]_{\eta>0}$. So, with a view to applying Property 1.1, we look for an estimation of the $L^\infty(Q)$-boundaries of u_η, independently from η. Moreover, to ensure that a Young measure associated with $(u_\eta)_{\eta>0}$ satisfies the constraint (1.3), it is necessary to establish an estimation of the penalized term in (1.5). Both results are obtained by introducing the viscous problem corresponding to $(\mathcal{P})_\eta$ and by using the convergence properties of viscous solutions to the weak entropy solution of $(\mathcal{P})_\eta$. So, in the rest of this paper, we assume:

(i) f (resp. g) is a $\mathcal{C}^2$-class function on $\mathbb{R}$ (resp. $\mathcal{C}^0$-class on $Q\times\mathbb{R}$). In addition one supposes that f (resp. g) is Lipschitzian on $\mathbb{R}$ (resp. Lipschitzian with respect to the third variable, uniformly in (t,x)).

(ii) θ and B_i, $i=1,\cdots,p$, belong to $W^{1,+\infty}(Q)$; $\theta\geq 0$ a.e. on Q.

(iii) $u^B\in L^\infty(\Sigma)$, $0\leq u^B\leq\theta$ in Σ, $u_0\in L^\infty(\Omega)$, $0\leq u_0\leq\theta(0,\cdot)$ in Q.

Particularly those hypotheses guarantee the existence of the quantity $M(t)$ defined for all real numbers t of $[0,T]$ by

$$M(t)=\|u_0\|_{L^\infty(\Omega)}e^{C_1 t}+\frac{C_2}{C_1}(e^{C_1 t}-1), \tag{2.1}$$

where C_1 is the sum of the Lipschitz constants of $g(t,x,u)$ and $f(u)\partial_{x_i}B_i(t,x)$ for all i of $\{1,\cdots,p\}$, with respect to the variable u.

Furthermore $C_2=\max\limits_{[0,T]\times\bar{\Omega}}|g(t,x,0)+f(0)\vec{\nabla}\cdot\vec{B}(t,x)|$.

Thus, according to the works of C.Bardos, A.Y.LeRoux & J.C.Nedelec[1], we have

Theorem 2.1. *For each value of the parameter $\eta>0$, the penalized problem $(\mathcal{P})_\eta$ has a unique weak entropy u_η in $BV(Q)\cap L^\infty(Q)$. This solution is the $L^p(Q)$ and $\mathcal{C}([0,T];L^p(\Omega))$ -limit, $1\leq p<+\infty$, when ϵ goes to 0^+, of the solutions' sequence $(u_{\epsilon,\eta})_{\epsilon>0}$ of the diffusion problems $\left[(\mathcal{P})_{\epsilon,\eta}\right]_{\epsilon>0}$ defined for each value of $\epsilon>0$ through the non homogeneous Dirichlet problem:*

Find $u_{\epsilon,\eta}$ in $H^1(Q)\cap L^\infty(Q)$, such that

$$\mathbb{H}(t,x,u_{\epsilon,\eta})+\frac{1}{\eta}\beta_\eta(t,x,u_{\epsilon,\eta})=\epsilon\Delta u_{\epsilon,\eta}\quad a.e.\ on\ Q, \tag{2.2}$$
$$u_{\epsilon,\eta}=u^B_\eta\ \text{ a.e. on }\Sigma,\ \ u_{\epsilon,\eta}(0,.)=u_{0,\eta}\ \text{ a.e. on }\Omega,$$

where u^B_η and $u_{0,\eta}$ are respectively the standard regularisations of u^B and u_0 by means of mollifer sequences indexed on the penalization parameter η. Moreover, β_η is obtained through β by changing θ in its spatial regularisation θ_η.

2.1. The Study of the Penalized Problem

Through an L^1-truncature method and by using the monotony of the penalized operator $\beta(t,x,u)$ with respect to u, we establish that the solution of $(\mathcal{P})_{\epsilon,\eta}$ verifies the maximum principle:

$$|u_{\epsilon,\eta}(t,x)|\leq M(t)\quad\text{a.e. in}\ \ Q, \tag{2.3}$$

where $M(t)$ is given by (2.1).

Now we look for an estimation of the penalized term in (2.2). Again, L^1-truncature arguments are developed by first considering the $L^2(Q)$-scalar product between $(\mathcal{E})_{\epsilon,\eta}$ and $\text{sign}_\lambda(-u_{\epsilon,\eta})\varphi$, where $\varphi \in \mathcal{C}^2_{c,+}(Q)$ and $\text{sign}_\lambda(\,\cdot\,)$, $\lambda > 0$, is the odd approximation of the function $\text{sign}(\cdot)$, defined for all positive real numbers x through $\text{sign}_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right)$.

Let $I_\lambda(u) = \int_0^u \text{sign}_\lambda(\tau)\,d\tau$. The Green formula leads to

$$\frac{1}{\eta}\int_Q (-u_{\epsilon,\eta})\text{sign}_\lambda(-u_{\epsilon,\eta})\varphi\,dxdt$$
$$= -\epsilon\int_Q \vec{\nabla}u_{\epsilon,\eta}\cdot\vec{\nabla}(\text{sign}_\lambda(-u_{\epsilon,\eta}))\,\varphi\,dxdt - \epsilon\int_Q \vec{\nabla}I_\lambda(-u_{\epsilon,\eta})\cdot\vec{\nabla}\varphi\,dxdt$$
$$+\int_Q I_\lambda(-u_{\epsilon,\eta})\partial_t\varphi\,dxdt - \int_Q \Big\{g(t,x,u_{\epsilon,\eta}) + f(0)\,\vec{\nabla}\cdot\vec{B}\Big\}\,\text{sign}_\lambda(-u_{\epsilon,\eta})\varphi\,dxdt$$
$$+\int_Q \{f(u_{\epsilon,\eta}) - f(0)\}\,\vec{B}\cdot\vec{\nabla}\varphi\,\text{sign}_\lambda(-u_{\epsilon,\eta})\,dxdt$$
$$+\int_Q \{f(u_{\epsilon,\eta}) - f(0)\}\,\vec{B}\cdot\vec{\nabla}\text{sign}_\lambda(-u_{\epsilon,\eta})\,\varphi\,dxdt.$$

Let us examine the right-hand side of this equality: the first term of the first line is negative (definition of $(\,\cdot\,)^-$), the second and third lines are bounded by a constant C_φ which only depends on φ (according to (2.3)) and the Sacks lemma shows that the fourth line tends to 0^+ when λ goes to 0^+. Then, after integrating by parts the second term of the first line and passing to the limit with λ, it comes that

$$\frac{1}{\eta}\int_Q |-u_{\epsilon,\eta}|\,\varphi\,dxdt \le C_\varphi + \epsilon\int_Q |-u_{\epsilon,\eta}|\,\Delta\varphi\,dxdt.$$

Next we consider the $L^2(Q)$-scalar product between the diffusion equation $(\mathcal{E})_{\epsilon,\eta}$ and $sgn_\lambda\Big([u_{\epsilon,\eta} - \theta_\eta]^+\Big)\varphi$. The arguments are the same as those developed previously. We especially use the majorations of the time and space derivatives of θ_η, independently from η, obtained by referring to its definition.

Then, there exists a constant C_φ which only depends on φ, such that

$$\frac{1}{\eta}\int_Q (u_{\epsilon,\eta} - \theta_\eta)^+\varphi\,dxdt \le C_\varphi + \epsilon\int_Q \Delta\theta_\eta\,\text{sign}\,(u_{\epsilon,\eta} - \theta_\eta)^+\,\varphi\,dxdt.$$

Hence, by adding the two previous estimates, it follows that

$$\tfrac{1}{\eta}\int_Q |\beta_\epsilon(t,x,u_{\epsilon,\eta})|\,\varphi dxdt \le C_\varphi + \epsilon\Big\{\int_Q \Delta\theta_\eta\,\text{sign}\,(u_{\epsilon,\eta} - \theta_\eta)^+\,\varphi + |-u_{\epsilon,\eta}|\,\Delta\varphi\Big\}dxdt,$$

where C_φ is a constant which only depends on φ.

The convergence properties recalled in Theorem 2.1 ensure, by passing to the limit when ϵ goes to 0^+, the following property.

Property 2.1. *For all strictly positive real numbers η,*

$$\|u_\eta\|_{L^\infty(Q)} \le M, \tag{2.4}$$

$$\forall\varphi \in \mathcal{C}^2_{c,+}(Q),\ \ \frac{1}{\eta}\,\|\beta_\eta(t,x,u_\eta)\varphi\|_{L^1(Q)} \le C_\varphi \tag{2.5}$$

where $M = M(T)$ and C_φ is a constant which only depends on φ.

Remark 2.1. Obviously, since u_η is the $BV(Q)$-weak entropy solution of $(\mathcal{P})_\eta$, it verifies the entropy inequality, for all real numbers k and for all positive functions ξ of $H_0^1(Q)$,

$$\int_Q \mathcal{L}^\eta(t,x,u_\eta,k,\xi)\, dxdt \geq 0,$$

where $\mathcal{L}^\eta(t,x,u,k,v) = \mathcal{L}(t,x,u,k,v) - \frac{1}{\eta}\beta_\eta(t,x,u)\ \operatorname{sign}(u-k)\,v.$

Besides, it fulfills the equivalent C.Bardos, A.Y.LeRoux & J.C.Nedelec form of the Dirichlet boundary condition : for all real numbers k and for all $L^1(\Sigma)$-positive function ζ,

$$\begin{aligned} &-\int_\Sigma \vec{F}(t,\sigma,u_\eta,k)\cdot\vec{n}\,\zeta\, d\mathcal{H}^p \\ &\leq \int_\Sigma \vec{F}\left(t,\sigma,u_\eta,u_\eta^B\right)\cdot\vec{n}\,\zeta\, d\mathcal{H}^p - \int_\Sigma \vec{F}\left(t,\sigma,k,u_\eta^B\right)\cdot\vec{n}\,\zeta\, d\mathcal{H}^p. \end{aligned} \tag{2.6}$$

However, to pass to the limit when η goes to 0^+, we must free ourselves from the parameter η in $\mathcal{L}^\eta$, by referring to the monotonicity of $\beta_\eta(t,x,.)$, and must consider some non linerarities continuous with respect to u_η. That is why some other specific properties of u_η have to be precised.

Firstly, in order to simply the writing, let us denote for all positive $\mathcal{C}^2(\mathbb{R})$-convex functions $\mathbb{E}$ such that $\mathbb{E}(0) = \mathbb{E}'(0) = 0$,

$$\begin{aligned} \vec{F_{\mathbb{E}}}(t,x,u,k) &= \int_k^u f'(\tau)\,\mathbb{E}'(\tau-k)\ d\tau\,\vec{B}(t,x), \\ G_{\mathbb{E}}^\eta(t,x,u,v) &= \mathbb{E}'(u-v)\Big[g(t,x,u) + \partial_t v + \frac{1}{\eta}\beta_\eta(t,x,u)\Big] \\ &\quad + \Big[\int_v^u \{f'(\tau)\,\vec{B}(t,x).\vec{\nabla}v + f(\tau)\,\vec{\nabla}\cdot\vec{B}(t,x)\}\mathbb{E}''(\tau-k)\ d\tau\Big], \\ \mathcal{L}_{\mathbb{E}}^\eta(t,x,u,k,v) &= \mathbb{E}(u-k)\partial_t v + \vec{F_{\mathbb{E}}}(t,x,u,k)\cdot\vec{\nabla}v - G_{\mathbb{E}}^\eta(t,x,u,k)\,v. \end{aligned}$$

Then, we have the next result:

Property 2.2. *For all real numbers k, for all positive functions ξ of $H_0^1(Q)$, the weak entropy solution of $(\mathcal{P})_\eta$ fulfills the inequality*

$$\int_Q \mathcal{L}_{\mathbb{E}}^\eta(t,x,u_\eta,k\theta_\eta,\xi)\,dxdt \geq 0. \tag{2.7}$$

Furthermore, u_η satisfies the Dirichlet boundary condition in the following sense, for all positive functions ζ of $L^1(\Sigma)$,

$$\begin{aligned} &-\int_\Sigma \vec{F}(t,\sigma,u_\eta,k\theta_\eta)\cdot\vec{n}\,\zeta d\mathcal{H}^p \\ &\leq \int_\Sigma \vec{F}\left(t,\sigma,u_\eta,u_\eta^B\right)\cdot\vec{n}\,\zeta d\mathcal{H}^p - \int_\Sigma \vec{F}\left(t,\sigma,k\theta_\eta,u_\eta^B\right)\cdot\vec{n}\,\zeta d\mathcal{H}^p \end{aligned} \tag{2.8}$$

Proof. Let us come back to the viscous problem $(\mathcal{P})_{\epsilon,\eta}$ related to $(\mathcal{P})_\eta$. By first taking into account the convexity of $\mathbb{E}$, we obtain the majoration a.e. on Q,

$$\epsilon\,\mathbb{E}'(u_{\epsilon,\eta}-k\theta_\eta)\,\Delta[u_{\epsilon,\eta}-k\theta_\eta] \leq \epsilon\,\Delta\mathbb{E}(u_{\epsilon,\eta}-k\theta_\eta).$$

Furthermore, the definition of $\vec{F_{\mathbb{E}}}(t,x,u,k\theta)$ allows us to turn the transport term into

$$\vec{\nabla}\cdot\left(f(u_{\epsilon,\eta})\vec{B}(t,x)\right)\mathbb{E}'(u_{\epsilon,\eta}-k\theta_\eta)=\vec{\nabla}\cdot\left(\vec{F_{\mathbb{E}}}(t,x,u_{\epsilon,\eta},k\theta_\eta)\right)$$
$$+k\int_{k\theta_\eta}^{u_{\epsilon,\eta}}\left\{f'(\tau)\vec{B}(t,x)\cdot\vec{\nabla}\theta_\eta+f(\tau)\vec{\nabla}\cdot\vec{B}(t,x)\right\}\mathbb{E}''(\tau-k\theta_\eta)\,d\tau.$$

Then, by multiplying the diffusion equation (2.2) with $\mathbb{E}'(u_{\epsilon,\eta}-k\theta_\eta)$, the next relation holds a.e. on Q:

$$\partial_t\mathbb{E}(u_{\epsilon,\eta}-k\theta_\eta)+\vec{\nabla}\cdot\left(\vec{F_{\mathbb{E}}}(t,x,u_{\epsilon,\eta},k\theta_\eta)\right)$$
$$+\mathbb{E}'(u_{\epsilon,\eta}-k\theta_\eta)\left[g(t,x,u_{\epsilon,\eta})+k(\partial_t\theta_\eta-\epsilon\Delta\theta_\eta)+\frac{1}{\eta}\beta_\eta(t,x,u_{\epsilon,\eta})\right]$$
$$+k\int_{k\theta_\eta}^{u_{\epsilon,\eta}}\left\{f'(\tau)\vec{B}(t,x)\cdot\vec{\nabla}\theta_\eta+f(\tau)\vec{\nabla}\cdot\vec{B}(t,x)\right\}\mathbb{E}''(\tau-k\theta_\eta)\,d\tau$$
$$\le\epsilon\Delta\mathbb{E}(u_{\epsilon,\eta}-k\theta_\eta).$$

We obtain the entropy inequality (2.7) by considering the $L^2(Q)$-scalar product between the above inequality and a positive test-function ξ element of $H_0^1(Q)$ and through passing to the limit when ϵ goes to 0^+ (by applying the convergence properties recalled in Theorem 2.1).

With a view to obtaining the boundary condition (2.8), we use the fact that, since u_η is a bounded function of bounded variation on Q, the inequality (2.6) holds for all real numbers k and for all $L^1(\Sigma)$-positive functions ζ. So, we consider a sequence of simple functions $(\theta_{\eta,m})_{m\in\mathbb{N}^*}$ defined for all m of $\mathbb{N}^*$ by $\theta_{\eta,m}=\sum\limits_{i=1}^{m}k_i\mathbb{I}_{\mathcal{B}_i}$, where k_i belongs to $\mathbb{R}$ for all i of $\{1,\cdots,m\}$ and $(\mathcal{B}_i)_{i\in\{1,\cdots,m\}}$ is a Borelian partition of Σ. By rewriting (2.6) with kk_i and $\mathbb{I}_{\mathcal{B}_i}\zeta$ respectively instead of k and ζ and by summing on i, we pass to the limit when m goes to infinity through the Lebesgue dominated convergence theorem, since the sequence $(\theta_{\eta,m})_{m\in\mathbb{N}^*}$ converges to θ_η in $L^1(Q)$ and a.e. on Q.

Indeed, by passing to the limit when η tends to 0^+, the inequality (2.7) will supply the entropy inequality (1.7) for a Young measure ν associated with $(u_\eta)_{\eta>0}$. However, (2.8) does not ensure that a Young measure trace on Σ related to ν (in the sense of Definition 1.2) fulfills the boundary condition (1.8). In fact, since $(u_\eta)_{\eta>0}$ is also a bounded sequence in $L^\infty(\Sigma)$, the relation (2.8) only gives some information about a Young measure on Σ corresponding to this sequence. That is why we need the next corollary.

Corollary 2.1. *Let u_η be the $BV(Q)\cap L^\infty(Q)$-weak entropy solution of $(\mathcal{P})_\eta$. Then for all real numbers k and for all positive functions ξ of $H^1(Q)$,*

$$\int_Q\mathcal{L}^\eta(t,x,u_\eta,k\theta_\eta,\xi)\,dxdt$$
$$\ge-\int_\Omega|u_{0,\eta}-k\theta_\eta(0,\cdot)|\,\xi(0,\cdot)\,dx+\int_\Sigma\vec{F}(t,\sigma,u_\eta,k\theta_\eta)\cdot\vec{n}\xi d\mathcal{H}^p.$$

Proof. First of all, since the entropy inequality (2.7) is fulfilled for all $\mathcal{C}^2(\mathbb{R})$-convex function $\mathbb{E}$ such that $\mathbb{E}(0)=\mathbb{E}'(0)=0$, then it is still satisfied when $\mathbb{E}$ is the term of a regular sequence approximating the absolute value function. At the limit we obtain for all real numbers k and for all functions ξ of $H_0^1(Q)$,

$$\int_Q \mathcal{L}^\eta\left(t,x,u_\eta,k\theta_\eta,\xi\right)\,dxdt \geq 0.$$

Now, the demonstration can be exactly developed as that of Theorem 1.1 (ii), but in a more regular context here because u_η is a bounded function with bounded variation on Q, it has a trace at $t=0$ and on Σ. Hence we choose in the previous entropy inequality the test-function $\xi(t,x) = \zeta(t,x)\,\phi(x)\,\varphi_m(t)$, where ζ belongs to $\mathcal{D}_+\left(\bar{Q}\right)$, ϕ is a $\mathcal{C}^2(\Omega)$-positive function with a compact support in Ω and φ_m is the Lipschtzian approximation of $\mathbb{I}_{]0,T[}$ already given by $\varphi_m(t) = \max\left(0,\min\left(mt, m\left(T-t\right),1\right)\right)$, for all m of $\mathbb{N}$. The definition of u_η's trace in $BV(Q)\cap L^\infty(Q)$ at $t=0$ permits to pass to the limit when m goes to infinity. It results in

$$\int_Q \mathcal{L}^\eta\left(t,x,u_\eta,k\theta_\eta,\zeta\phi\right)\,dxdt + \int_\Omega \zeta\left(0,\cdot\right)\phi\left|u_{0,\eta}-k\theta_\eta\left(0,\cdot\right)\right|\,dx \geq 0.$$

Now, for each value of the parameter $\delta>0$, we introduce the function ρ_δ element of $\mathcal{C}^2\left(\bar{\Omega}\right)$, which verifies the following properties[7]:

$$\begin{cases} \rho_\delta \equiv 1 \text{ on } \Gamma, \quad \rho_\delta \equiv 0 & \text{on } \left\{x\in\Omega,\ \operatorname{dist}\left(x,\Gamma\right)\geq\delta\right\}, \\ 0\leq\rho_\delta\leq 1 & \text{on } \Omega, \quad \left\|\vec{\nabla}\rho_\delta\right\|_{L^\infty(\Omega)^p} \leq \frac{C^{st}}{\delta}, \end{cases}$$

where C^{st} is a constant independent of δ. Then, by taking ϕ equal to $1-\rho_\delta$, we have

$$\int_Q \mathcal{L}^\eta\left(t,x,u_\eta,k\theta_\eta,\zeta\right)\left(1-\rho_\delta\right)dxdt - \int_Q \vec{F}\left(t,x,u_\eta,k\theta_\eta\right)\cdot\vec{\nabla}\rho_\delta\,\zeta\,dxdt$$
$$+\int_\Omega \zeta\left(0,\cdot\right)\left(1-\rho_\delta\right)\left|u_0-k\theta_\eta\left(0,\cdot\right)\right|dx \geq 0.$$

As u_η belongs to $BV(Q)\cap L^\infty(Q)$, we transform the second integral in the above inequality through the Green formula. As the sequence $\left(\rho_\delta\right)_{\delta>0}$ converges $d\mathbb{M}$-a.e. to 0, where $d\mathbb{M}$ is the bounded Radon measure associated with the distribution $\vec{\nabla}\cdot\vec{F}\left(t,x,u_\eta,k\theta_\eta\right)$, then it is possible to take the δ-limit in the previous inequality to obtain, for all functions ζ of $\mathcal{D}_+\left(\bar{Q}\right)$,

$$\int_Q \mathcal{L}^\eta\left(t,x,u_\eta,k\theta_\eta,\zeta\right)dxdt \geq \int_\Sigma \vec{F}\left(t,\sigma,u_\eta,k\theta_\eta\right)\cdot\vec{n}\zeta d\mathcal{H}^p - \int_\Omega \zeta\left(0,\cdot\right)\left|u_{0,\eta}-k\theta_\eta\right|dx.$$

To complete the proof, we refer to the density of $\mathcal{D}\left(\bar{Q}\right)$ into $H^1(Q)$.

2.2. Existence of an Entropy Measure-Valued Solution

Given that $\left(u_\eta\right)_{\eta>0}$ is a bounded sequence in $L^\infty(Q)$, the reminders of Property 1.1, the a priori estimates developed in Property 2.1 and the entropy relations established in Property 2.2 will enable us to specify the behavior of the sequence $\left(u_\eta\right)_{\eta>0}$ when η goes to 0^+. We precisely have

Theorem 2.1. *There exists at least an emvs to* (1.3) *and* (1.4).

Proof. Let k be an element of $[0,1]$. As in the regularized entropy inequality (2.7) the penalized term is negative, then one may take the η-limit by referring to the $L^\infty(Q)$ weak-$*$ convergence properties. This provides us with the inequality (1.7) by coming back to the particular choice of Kruskov's entropy pairs. Moreover, the penalized term's estimate (2.5) ensures that a Young measure ν associated with the sequence $(u_\eta)_{\eta>0}$ fulfills the obstacle condition (1.6). Finally ν is an emvs to (1.3) and (1.4) in the sense of Definition 1.1.

Let ν be a Young measure related to $(u_\eta)_{\eta>0}$. According to Properties 1.2 and 1.3, since ν is an emvs to (1.3) and (1.4), one may give a sense to an initial condition in $L^1(\Omega)$ and specify the behavior of ν in the neighborhood of Γ. Thus an entropy relation is given for ν, through which, by taking suitable test-functions and by following the same kind of calculuses as those already developed in Properties 1.2 and 1.3, we are able to prove

Theorem 2.2. *This emvs ν is an EMVS to the problem $(\mathcal{P})$.*

Proof. We have to check that the boundary conditions (1.8) and (1.9) are satisfied. For this purpose, we take the η-limit in the inequality established in Corollary 2.1. According to the reminders of Property 1.1, for all real numbers k of $[0,1]$ and for all positive functions ξ of $H^1(Q)$, we have

$$\int_{Q\times\mathbb{R}}\{|\lambda-k\theta|\,\partial_t\xi+\vec{F}(t,x,\lambda,k\theta)\cdot\vec{\nabla}\xi-\chi_k(t,x)\,\xi\}\,d\nu_{(t,x)}(\lambda)\,dxdt$$
$$\geq\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\sigma,\lambda,k\theta)\cdot\vec{n}\;\xi\,d\tilde{\mu}(t,\sigma,\lambda)-\int_{\Omega}|u_0-k\theta(0,\cdot)|\,\xi(0,\cdot)\,dx,$$

where $\chi_k(t,x)$ is the $L^\infty(Q)$ weak-$*$ limit of the bounded non continuous function $G(t,x,u_\eta,k\theta_\eta)$ and $\tilde{\mu}$ is a Young measure on Σ associated with the sequence of u_η's trace on Σ.

Firstly, let ζ be an element of $\mathcal{D}_+(\Omega)$ and $m\in\mathbb{N}$. In the previous inequality we choose the test-function $\xi(t,x)=\zeta(x)\max(0,\min(1-mt,1))$. Using the result established in Property 1.2 allows to take the limit when m goes to infinity. So one has

$$\lim_{t\to0^+}\text{ess}\int_{\Omega\times\mathbb{R}}|\lambda-k\theta|\,\zeta\,d\nu_{(t,x)}(\lambda)\,dx\leq\int_{\Omega}|u_0-k\theta(0,\cdot)|\,\zeta dx.$$

Therefore by replacing the real parameter k by the function $k^*(.)$ defined a.e. on Ω by

$$k^*(x)=\begin{cases}\frac{u_0(x)}{\theta(0,x)}, & \text{if }\theta(0,x)\neq0,\\ 0, & \text{else,}\end{cases}$$

as shown in the demonstration of Property 1.2, one proves that the emvs ν satisfies the initial condition (1.9).

Secondly, let ζ be an element of $\mathcal{D}_+(\bar{Q})$ and $m\in\mathbb{N}$. We consider now the test-function $\xi(t,\bar{x})=\zeta(t,\sigma)\max(0,\min(1-m\tau,1))$ with $\bar{x}=x_{(\sigma,\tau)}$ in the variables (σ,τ) introduced in Definition 1.2. By taking the limit when m goes to infinity (Property 1.3), we have for all real numbers k of $[0,1]$,

$$\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\bar{x},\lambda,k\theta)\cdot\vec{n}\zeta d\tilde{\mu}(t,\sigma,\lambda)\leq\int_{\Sigma\times\mathbb{R}}\vec{F}(t,\bar{x},\lambda,k\theta)\cdot\vec{n}\zeta\,d\mu(t,\sigma,\lambda).$$

Then, as shown in Properties 1.2 and 1.3, the real parameter k can be replaced by the function $k^*(\cdot,\cdot)$ defined a.e. on Σ by

$$k^*(t,\sigma)=\begin{cases}\frac{u^B(\sigma)}{\theta(t,\sigma)}, & \text{if }\theta(t,\sigma)\neq0,\\ 0, & \text{else.}\end{cases}$$

This leads to the same inequality as before with u^B instead of $k\theta$. Now, since u_η is the BV-weak entropy solution $(\mathcal{P})_\eta$, it fulfills the boundary condition (2.8) established in Property

2.2. So, by passing to the limit when η goes to infinity and by using the two previous relations between $\tilde{\mu}$ and μ, we prove that the emvs ν verifies the boundary condition (1.8).

§3. Uniqueness of the Entropy Measure-Valued Solution

Now we establish the uniqueness of an EMVS to the problem $(\mathcal{P})$. As has been specified in the introduction, this uniqueness result first ensures that the family of probability measures $\left(\nu_{(t,x)}\right)_{(t,x)\in Q}$ related to $(u_\eta)_{\eta>0}$ has a punctual support. Namely, there exists a bounded and measurable function u on Q such that, for a.e. (t,x) in Q, $\nu_{(t,x)} = \delta_{u(t,x)}$, where δ_Z is the Dirac mass centered on the point Z. The function u thus defined will be called the Entropy Solution of $(\mathcal{P})$. Moreover, it ensures that the whole approximating sequence $(u_\eta)_{\eta>0}$ converges to u in $L^q(Q)$, $1 \le q < +\infty$ and a.e. in Q.

3.1. The Kruskov Relation

The uniqueness proof is classical: it uses that developed by S.N.Kruskov[7] through the splitting in two of the time and space variables. However, we will remark that the introduction of a forced bilateral obstacle condition gives rise to the choice of an entropy family depending on (t,x), through the function θ. Without adding many technical difficulties, this dependence on time and space entails many additional calculuses and leads us to refer to the notion of an Entropy Process Solution introduced by R.Eymard, T.Gallouet & R.Herbin[6] through the next statement.

Property 3.1. *Let $\mathcal{O}$ be an open subset of $\mathbb{R}^{p+1}$ and let $w \to \nu_w$ a Young measure with support in $[-M,M]$. Then, there exists a function π in $L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous bounded functions ψ on $\mathcal{O}\times[-M,M]$,*

$$\int_{\mathbb{R}\times\mathcal{O}} \psi(x,\lambda)\, d\nu_w(\lambda)\, dx = \int_{]0,1[\times\mathcal{O}} \psi(x,\pi(\alpha,w))\, d\alpha dx \ \ fora.e.\ w\ in\ \mathcal{O}.$$

The demonstration is based on the properties of the generalized inverse of the distribution function linked to a probability measure. Hence the integration with respect to the measure $d\nu_w$ is turned into an integration with respect to the Lebesgue measure on $]0,1[$.

Thus, the fine Properties 1.2, 1.3 and Definition 1.2, given for an emvs to (1.3) and (1.4), can be expressed by using property 3.1. This leads to the notion of an Entropy Process *Solution* to (1.3) and (1.4). To begin with, we state the next result.

Theorem 3.1. *Let us assume that ν and ϖ are two EMVS to the problem $(\mathcal{P})$ related respectively to the initial and the boundary conditions u_0, u^B and $\hat{u}_0$, $\hat{u}^B$. Then for a.e. t in $]0,T[$,*

$$\begin{aligned}&\int_{\mathbb{R}^2\times\Omega} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\, dx\\ &\le \Big(2M'_f \left\|\vec{B}\right\|_\infty \int_0^t \int_\Gamma \left|u^B-\hat{u}^B\right| d\mathcal{H}^p ds + \int_\Omega |u_0-\hat{u}_0|\, dx\Big) e^{M'_g t},\end{aligned} \tag{3.1}$$

where M'_f and M'_g are respectively the Lipschitz constant of f and g (with respect to the third variable, uniformly in (t,x)).

Proof. Let m be an element of $\mathbb{N}^*$ and let ρ_m be the standard mollifer sequence in $\mathbb{R}^m$. For each n of $\mathbb{N}^*$, let us denote by $(\rho_{m,n})_{n\in\mathbb{N}}$ the sequence defined by

$$\rho_{m,n}(x) = n^m \rho_m(nx).$$

Lastly, $\mathcal{E}$ will stand for the open subset $]0,1[\times Q$.

Let us now consider two entropy process solutions π and ω associated respectively with the EMVS ν and ϖ. We introduce the functions k^* and k of $L^\infty(\mathcal{E}_3)$,

$$k(\beta,s,y) = \begin{cases} \frac{\omega(\beta,s,y)}{\theta(s,y)}, & \text{if } \theta(s,y)\neq 0, \\ 0 \text{ else.} \end{cases}, \quad k^*(\alpha,t,x) = \begin{cases} \frac{\pi(\alpha,t,x)}{\theta(t,x)}, & \text{if } \theta(t,x)\neq 0, \\ 0, & \text{else.} \end{cases}$$

In this way, $0\le k\le 1$ and $0\le k^*\le 1$ a.e. on $\mathcal{E}$. Moreover

$$k(\beta,s,y)\theta(s,y) = \omega(\beta,s,y) \text{ and } k^*(\alpha,t,x)\,\theta(t,x) = \pi(\alpha,t,x) \text{ a.e. on } \mathcal{E}.$$

Let us consider the function Λ defined on $Q\times Q$ through

$$\Lambda(t,x,s,y) = \varphi(t,x)\,\rho_{p,n}(x-y)\,\rho_{1,n}(t-s),$$

where φ is an element of $\mathcal{D}_+\left(]0,T[\times\bar{\Omega}\right)$.

In the entropy relation (1.10) satisfied by π one chooses $k=k(\beta,s,y)$, the test-function being equal to $\Lambda(\cdot,\cdot,s,y)\theta(s,y)$. We make sure that it is still possible to integrate with respect to the Lebesgue measure $d\beta dyds$ on $\mathcal{E}$. The same reasoning leads us to choose a test-function equal to $\Lambda(t,x,\cdot,\cdot)\theta(t,x)$ and $k=k^*(\alpha,t,x)$, in the inequality (1.10) fulfilled by ω and written in the variables (β,s,y). Then, we integrate over $\mathcal{E}$ with respect to the measure $d\alpha dxdt$.

Then, the two expressions obtained above are added up and taking into account the definition of Λ the partial derivatives in s, t, x and y are developed. It follows that

$$\sum_{i=1}^{3} I_{i,n} \ge -I_{4,n} - I_{5,n} + I_{6,n}, \text{ with}$$

$$
\begin{aligned}
I_{1,n} = & \int_{\mathcal{E}\times\mathcal{E}} \tilde{\Delta}\left(\mathrm{sign}, \pi, \omega\right) \{\tilde{\Delta}\left(id, \pi, \omega\right) \partial_t \varphi - \left(\tilde{\Delta}\left(f \vec{\nabla} \cdot \vec{B}, k\theta, k^* \theta\right) + \tilde{\Delta}\left(g, \pi, \omega\right)\right) \varphi \\
& + \left(\tilde{\Delta}\left(f\left(\cdot\right) \vec{B}, \pi, \omega\right) - \tilde{\Delta}\left(f\left(\cdot\right) \vec{B}, k\theta\left(t,x\right), \omega\right)\right) \cdot \vec{\nabla}\varphi\} \rho_{p,n} \rho_{1,n}\, d\mathcal{L}, \\
I_{2,n} = & \int_{\mathcal{E}\times\mathcal{E}} \tilde{\Delta}\left(\mathrm{sign}, \pi, \omega\right) \{[\pi \partial_s \theta(s,y) - \omega \partial_t \theta(t,x)] \\
& + \left[f'\left(k^* \theta(s,y)\right) \vec{B}(s,y) \cdot \vec{\nabla}\theta(s,y)\, \pi - f'\left(k\theta(t,x)\right) \vec{B}(t,x) \cdot \vec{\nabla}\theta(t,x)\, \omega\right] \} \Lambda d\mathcal{L}, \\
I_{3,n} = & \int_{\mathcal{E}\times\mathcal{E}} \tilde{\Delta}\left(\mathrm{sign}, \pi, \omega\right) \{\tilde{\Delta}\left(f\left(\cdot\right) \vec{B}, \pi, k^* \theta\left(s,y\right)\right) \\
& - \tilde{\Delta}\left(f\left(\cdot\right) \vec{B}, k\theta\left(t,x\right), \nu\right)\} \cdot \vec{\nabla}\rho_{p,n} \rho_{1,n} \varphi d\mathcal{L}, \\
I_{4,n} = & \int_{\Omega^2 \times]0,1[\times]0,T[} \left|u_0\left(y\right) - k^* \theta\left(0,y\right)\right| \theta\left(t,x\right) \Lambda\left(t,x,0,y\right)\, d\alpha dx dt dy, \\
I_{5,n} = & \int_{\Sigma\times\mathcal{E}} \vec{F}\left(t, \sigma, \tilde{\pi}\left(\alpha, t, \sigma\right), u^B\right) \cdot \vec{n}\, \theta\left(s,y\right) \Lambda\left(t,\sigma,s,y\right) d\alpha d\mathcal{H}^p_{(t,\sigma)} dy ds \\
& + \int_{\Sigma\times\mathcal{E}} \vec{F}\left(s, \tau, \tilde{\omega}\left(\beta, s, \tau\right), \hat{u}^B\right) \cdot \vec{n}\, \theta\left(t,x\right) \Lambda\left(t,x,s,\tau\right) d\beta d\mathcal{H}^p_{(s,\tau)} dx dt, \\
I_{6,n} = & \int_{\Sigma\times\mathcal{E}} \vec{F}\left(t, \sigma, k\left(\beta, s, y\right) \theta\left(t,\sigma\right), u^B\right) \cdot \vec{n}\, \theta\left(s,y\right) \Lambda\left(t,\sigma,s,y\right) d\beta d\mathcal{H}^p_{(t,\sigma)} dy ds \\
& + \int_{\Sigma\times\mathcal{E}} \vec{F}\left(s, \tau, k^*\left(\alpha, t, x\right) \theta\left(s,\tau\right), \hat{u}^B\right) \cdot \vec{n}\, \theta\left(t,x\right) \Lambda\left(t,x,s,\tau\right) d\alpha d\mathcal{H}^p_{(s,\tau)} dx dt.
\end{aligned}
$$

To clarify, we have denoted

$$\tilde{\Delta}\left(f, u, v\right) = f\left(t, x, u\left(t,x\right)\right) \theta\left(s,y\right) - f\left(s, y, v\left(s,y\right)\right) \theta\left(t,x\right)$$

and $d\mathcal{L} = d\alpha dx dt\, d\beta dy ds$. Moreover, in $I_{5,n}$ the functions $\tilde{\pi}$ and $\tilde{\omega}$ represent the "trace processes" related respectively to π and ω; namely, the processes corresponding to the Young measure traces on Σ related to ν and ϖ, in the sense of Definition 1.2.

We seek to calculate the limit of each integral $I_{i,n}$, when n goes to infinity. The argumentation focuses on the notion of Lebesgue points for an integrable function and uses the Lipschitzian properties of the non linearities. Besides, the order of all integrations can be permuted. First of all, we state

Lemma 3.1. *Let u and v be two elements of $L^\infty\left(Q\right)$, ϕ a locally Lipschitzian function on $Q \times \mathbb{R}$ and ξ a bounded function on Q. We denote*

$$\Theta\left(t,x,s,y\right) = \rho_{p,n}\left(x-y\right) \rho_{1,n}\left(t-s\right) \xi\left(t,x\right)$$

and we consider

$$I_n = \int_{Q\times Q} \mathrm{sign}\left(u\left(t,x\right)\theta\left(s,y\right) - v(s,y)\theta(t,x)\right) \tilde{\Delta}\left(\phi, u, v\right) \Theta dx dt dy ds.$$

Then, $\lim\limits_{n\to+\infty} I_n = \int_Q \Delta\left(\phi\right) \theta \xi\ dxdt$*, where, in order to clarify, we set*

$$\Delta\left(\phi\right) = \mathrm{sign}\left(u\left(t,x\right) - v\left(t,x\right)\right) \{\phi\left(t,x,u\left(t,x\right)\right) - \phi\left(t,x,v\left(t,x\right)\right)\}.$$

Proof. We define $\mathcal{I}(a,b) = [\min(a,b); \max(a,b)]$ and we factorize

$$\begin{aligned}&\operatorname{sign}(u\theta(s,y) - v\theta(t,x))\,\tilde{\Delta}(\phi,u,v)\\ &= \operatorname{sign}(u\theta(s,y) - v\theta(t,x))\,[\phi(t,x,v) - \phi(s,y,v)]\,\theta(t,x)\\ &\quad + \operatorname{sign}(u\theta(s,y) - v\theta(t,x))\,[\theta(s,y) - \theta(t,x)]\,\phi(t,x,v)\\ &\quad + \operatorname{sign}(u\theta(s,y) - v\theta(t,x))\,\left[\phi(t,x,u) - \phi(t,x,v)\right]\,\theta(s,y).\end{aligned}$$

Since ϕ is locally Lipschitzian on $Q \times \pi_T$ and as θ belongs to $W^{1,+\infty}(Q)$, the integrals relative to the first line and to the second line tend to 0. For the third line l_3, whether or not $u(t,x)\,\theta(s,y)$ is an element of $\mathcal{I}(v(s,y)\,\theta(t,x); v(t,x)\,\theta(s,y))$, one establishes the existence of a constant C, such that a.e. on $Q \times Q$,

$$|l_3 - \theta(s,y)\,\Delta(\phi)| \le C\left(|v(s,y) - v(t,x)| + |\theta(s,y) - \theta(t,x)|\right).$$

Hence, according to the continuity of θ and to the definition of the Lebesgue points of v, I_n has got the same limit as

$$\int_{Q\times Q} \theta(s,y)\,\Delta(\phi)\,\xi(t,x)\;dxdtdyds.$$

This limit is calculated by referring to the continuity of θ on Q, which completes the proof of Lemma 3.1.

Lemma 3.1 is used, $u(t,x)$ and $v(s,y)$ being changed respectively into $\pi(\alpha,t,x)$ and $\omega(\beta,s,y)$, for a.e. (α,β) in $]0,1[^2$. Then, substitute $\phi(\cdot,\cdot,\cdot)$ and $\xi(\cdot,\cdot)$ in the statement of Lemma 3.1 with their mathematical expressions in each line of $I_{1,n}$. Furthermore, according to the regularity of $f(\cdot)\,\vec{B}$ and the continuity of θ, it is clear that the integral conming from the quantities $\tilde{\Delta}\left(f(\cdot)\,\vec{B}, k\theta(t,x), \omega\right)$ or $\tilde{\Delta}\left(f(\cdot)\,\vec{B}, \pi, k^*\theta(s,y)\right)$ tends to 0. The same remark is valid when $f(\cdot)\,\vec{B}$ is turned into $f(\cdot)\,\vec{\nabla}\cdot\vec{B}$, which gives us the limit of the integral arising from $\tilde{\Delta}\left(f(\cdot)\,\vec{B}, k\theta, k^*\theta\right)$. Hence we have

$$\lim_{n\to+\infty} I_{1,n} = \int_{Q\times]0,1[^2} (\partial_t\varphi\Delta(id_{\mathbb{R}}) + \Delta(f)\,\vec{B}\cdot\vec{\nabla}\varphi + \Delta\left(f(\cdot)\;\vec{\nabla}\cdot\vec{B} - g\right)\varphi)\theta\,d\alpha d\beta dxdt. \quad (3.2)$$

For the study of $I_{2,n}$, let us come back to the demonstration of Lemma 3.1 and assume that in $\tilde{\Delta}(\phi,u,v)$ the function θ is turned into its spatial derivative. Then the quantity $\partial_t\theta(s,y)\,\Delta(\phi)$ is subtracted from the third line. We take advantage of the fact that for a.e. (t_0,x_0) in Q such that $\theta(t_0,x_0) = 0$, one has $\partial_t\theta(t_0,x_0) = 0$. That way a.e. on $\mathcal{E}_3 \times \mathcal{E}$

$$\operatorname{sign}(u(t,x) - v(t,x))\,\partial_t\theta(s,y) = \operatorname{sign}(u(t,x)\,\theta(s,y) - v(t,x)\,\theta(s,y))\,\partial_t\theta(s,y),$$

and the arguments are now the same as previously. This reasoning being also valid when $\partial_t\theta$ is changed into $\vec{\nabla}\theta$, we finally obtain

$$\lim_{n\to+\infty} I_{2,n} = \int_{Q\times]0,1[^2} \left(\vec{B}\cdot\vec{\nabla}\theta\,\Delta(f'(\cdot)\;id_{\mathbb{R}}) + \partial_t\theta\Delta(id_{\mathbb{R}})\right)\varphi\,d\alpha d\beta dxdt. \quad (3.3)$$

To study the limit of $I_{3,n}$, we must compensate for the term $\vec{\nabla}\rho_{p,n}$ through an expresssion like $(t-s)\,\epsilon(t-s)$ or $(y-x)\,\tilde{\epsilon}(y-x)$, where ϵ and $\tilde{\epsilon}$ are integrable functions such that $\epsilon(0) = \tilde{\epsilon}(0) = 0$. Hence, when the quantity $|t-s|$ or $|y-x|$ is in the order of $\frac{1}{n}$, some estimates in the order of n^{p+1} are given. Then, the limit with respect to n can be calculated by referring to the notion of Lebesgue points on Q. So, if h is an integrable function on Q, in all the Lebesgue points (t,x) of h,

$$\lim_{n\to+\infty}\frac{1}{n}\int_{Q\times Q}|h(t,x)-h(s,y)|\left|\vec{\nabla}\rho_{p,n}\left(x-y\right)\right|\rho_{1,n}(t-s)\,dxdtdyds=0. \tag{3.4}$$

Hence, the regularity of f and $\vec{B}$ is used to divide the integrated term in $I_{3,n}$ into the five vector-valued functions :

$$\vec{L}_1=\left\{f\left(\omega\right)\theta\left(t,x\right)-f\left(k^*\theta(s,y)\right)\theta\left(s,y\right)\right\}\sum_{j=1}^{p}\int_{x_j}^{y_j}\partial_{x_j}\vec{B}\left(t,\lambda_j\right)d\lambda,$$

$$\vec{L}_2=\left\{f\left(k^*\theta\left(s,y\right)\right)\theta\left(s,y\right)-f\left(\omega\right)\theta\left(t,x\right)\right\}\int_s^t\partial_t\vec{B}\left(\tau,x\right)d\tau,$$

$$\vec{L}_3=\left\{\theta\left(t,x\right)-\theta\left(s,y\right)\right\}\left\{f'\left(\pi\right)k^*\theta\left(s,y\right)-f'\left(k\theta\left(t,x\right)\right)k\theta\left(t,x\right)\right\}\vec{B}\left(t,x\right),$$

$$\vec{L}_4=\left\{\theta\left(t,x\right)-\theta\left(s,y\right)\right\}\left\{f\left(k^*\theta\left(s,y\right)\right)\vec{B}\left(t,x\right)-f\left(k\theta\left(t,x\right)\right)\vec{B}\left(s,y\right)\right\},$$

$$\vec{L}_5=\left\{\theta\left(t,x\right)-\theta\left(s,y\right)\right\}^2\Big(\theta\left(s,y\right)\left(k^*\right)^2f''\left(\zeta_1\right)+\theta\left(t,x\right)k^2f''\left(\zeta_2\right)\Big)\vec{B}\left(t,x\right)$$

with

$$\zeta_1=\zeta\pi+\left(1-\zeta\right)k^*\theta\left(s,y\right),\ \zeta_2=\zeta'\omega+\left(1-\zeta'\right)k\theta\left(t,x\right),\ \ \left(\zeta',\zeta\right)\in\left]0,1\right[^2,$$

and by denoting for any real number λ,

$$\lambda_k=\left(y_1,\cdots,y_{k-1},\lambda,x_{k+1},\cdots,x_p\right).$$

Firstly, according to the technical point (3.4) and since f is a locally Lipschitzian function on $\mathbb{R}$, the integral given by $\vec{L}_1$ has got the same limit as that arising from

$$\vec{L}_1^*=\left\{f\left(\omega\left(\beta,s,y\right)\right)\theta\left(t,x\right)-f\left(\pi\left(\alpha,t,x\right)\right)\theta\left(s,y\right)\right\}\sum_{j=1}^{p}\int_{x_j}^{y_j}\partial_{x_j}\vec{B}\left(t,\lambda_j\right)d\lambda.$$

The arguments already developed to study the second and the third line of the decomposition in Lemma 3.1 prove that the integral coming from $\vec{L}_1^*$ has got the same limit as

$$\int_{\mathcal{E}\times\mathcal{E}}\theta\left(s,y\right)\Delta\left(f\right)\sum_{j=1}^{p}\int_{x_j}^{y_j}\partial_{x_j}\vec{B}\left(t,\lambda_j\right)d\lambda\cdot\vec{\nabla}\rho_{p,n}\rho_{1,n}\varphi d\mathcal{L}.$$

Given that $\vec{\nabla}_x\rho_{p,n}\left(x-y\right)=-\vec{\nabla}_y\rho_{p,n}\left(x-y\right)$, an integration by parts is taken. Then there is no difficulty in passing to the limit with n. So the term given by $\vec{L}_1$ tends to

$$-\int_{Q\times]0,1[^2}\vec{\nabla}\cdot\vec{B}\,\Delta\left(f\right)\,\theta\varphi\,d\alpha d\beta dxdt.$$

Similarly, the integral coming from $\vec{L}_2$ goes to 0.

In so far as θ belongs to $W^{1,+\infty}\left(Q\right)$, we make sure that the integral corresponding to $\vec{L}_5$ tends to 0. Moreover, in order to determine the limit of that given by $\vec{L}_3$, we first remark that the integrated quantity can also be written as

$$\operatorname{sign}\left(k^*-k\right)\left\{\theta\left(t,x\right)-\theta\left(s,y\right)\right\}\left\{f'\left(\pi\right)\ k^*\theta\left(s,y\right)-f'\left(k\theta\left(t,x\right)\right)\ k\theta\left(t,x\right)\right\}\vec{B}\left(t,x\right).$$

By judging whether or not $k^*\left(\alpha,t,x\right)$ is an element of $\mathcal{I}\left(k\left(\beta,t,x\right);k\left(\beta,s,y\right)\right)$ we resort to the technical result (3.4) so as to establish that the integral arising from $\vec{L}_3$ has got the same limit as

$$\int_{\mathcal{E}\times\mathcal{E}} \{\theta(t,x)-\theta(s,y)\}\,\Delta\left(f'(\cdot)\,id_{\mathbb{R}}\right)\vec{B}\cdot\vec{\nabla}\rho_{p,n}\rho_{1,n}\varphi\,d\mathcal{L}.$$

Then, we take an integration by parts with respect to the variable y. These arguments are used again to study the integral stemming from $\vec{L}_4$.

Thereafter, we get

$$\lim_{n\to+\infty} I_{3,n} = \int_{Q\times]0,1[^2} \left(\vec{\nabla}\theta\cdot\vec{B}\Delta\left(f-f'\,id_{\mathbb{R}}\right)-\theta\vec{\nabla}\cdot\vec{B}\,\Delta(f)\right)\varphi d\alpha d\beta dx dt. \tag{3.5}$$

Lastly, to complete the proof of the Kruskov relation (3.1) we now just have to focus on the boundary terms $I_{4,n}$, $I_{5,n}$ and $I_{6,n}$.

As to the first one, since φ belongs to $\mathcal{D}\left(]0,T[\times\bar{\Omega}\right)$, for n large enough, the quantity $\varphi(t,x)\,\rho_{1,n}(t)$ is equal to 0 for all real numbers t of $[0,T]$. So

$$\lim_{n\to+\infty} I_{4,n}=0. \tag{3.6}$$

For the other, since θ belongs to $\mathcal{C}\left(\bar{\Omega}\right)$, $\int\limits_Q \theta(s,y)\,\rho_{p,n}(\sigma-y)\,\rho_{1,n}(t-s)\,dyds$ converges to $\frac{1}{2}\theta(t,\sigma)$ on Σ and $\int\limits_Q \rho_{p,n}(x-\tau)\,\rho_{1,n}(t-s)\,\theta(t,x)\,\varphi(t,x)\,dxdt$ converges to $\frac{1}{2}\theta(s,\tau)\,\varphi(s,\tau)$ on Σ, as n goes to infinity. Hence, the Lebesgue dominated convergence theorem gives

$$\lim_{n\to+\infty} I_{5,n} = \frac{1}{2}\int_{\Sigma\times]0,1[} \left(\vec{F}\left(t,\sigma,\tilde{\pi},u^B\right)+\vec{F}\left(t,\sigma,\tilde{\omega},u^B\right)\right)\cdot\vec{n}\,\theta\varphi\,d\alpha d\mathcal{H}^p. \tag{3.7}$$

In order to determine the limit of $I_{6,n}$, let us come back to the change of variables introduced in Definition 1.2: ϵ being a strictly positive real number, for y in the neighborhood of Γ, we set $y=\tau-\kappa\vec{n}(\tau)$, where (τ,κ) belongs to $\Gamma\times]0,\epsilon[$. If we denote by $J(\tau,\kappa)$ the Jacobian determinant associated with this change of coordinates and $\bar{y}=y_{(\tau,\kappa)}$, we are led to transform the first line of $I_{6,n}$ into

$$\begin{aligned} I'_{6,n} = \int_{\Sigma\times]0,1[\times\Sigma\times]0,\epsilon[} & \vec{F}\left(t,\sigma,k(\beta,s,\bar{y})\,\theta(t,\sigma)\,,u^B(t,\sigma)\right)\cdot\vec{n}\,\theta(s,\bar{y}) \\ & \times\Lambda(t,\sigma,s,\bar{y})\,d\beta d\mathcal{H}^p_{(t,\sigma)}\,J(\tau,\kappa)\,d\mathcal{H}^p_{(s,\tau)}d\kappa. \end{aligned}$$

Let δ be a strictly positive real number. Given the Lusin theorem, there exists a continuous function $\tilde{u}^B$ on Σ and a Borelian subset $\mathcal{K}$ of Σ, with $d\mathcal{H}^p(\Sigma\backslash\mathcal{K})\le\delta$, such that $u^B=\tilde{u}^B$ a.e. on $\mathcal{K}$.

As f and θ are bounded functions, there exists a constant C^{st} such that

$$\left|I'_{6,n}-I''_{6,n}\right|\le C^{st}\,\delta,$$

where $I''_{6,n}$ is deduced from $I'_{6,n}$ by changing u^B into $\tilde{u}^B$. Furthermore, in so far as $\vec{F}(t,\sigma,k(\beta,s,\bar{y})\theta(t,\sigma),\tilde{u}^B(t,\sigma))\theta(s,\bar{y})\varphi(t,\sigma)$ is uniformly continuous on the compact support of φ, $I''_{6,n}$ has got the same limit, when n goes to infinity, as

$$\begin{aligned} I'''_{6,n} = \int_{\Sigma\times]0,1[\times\Sigma\times]0,\epsilon[} & \vec{F}\left(s,\bar{y},\omega(\beta,s,\bar{y})\,,\tilde{u}^B(t,\bar{y})\right)\cdot\vec{n}\,\theta(s,\bar{y}) \\ & \times\Lambda(t,\sigma,s,\bar{y})\,d\beta d\mathcal{H}^p_{(t,\sigma)}\,J(\tau,\kappa)\,d\mathcal{H}^p_{(s,\tau)}d\kappa. \end{aligned}$$

Then, by using a local map method[14] one shows that

$$\lim_{n\to+\infty} I'''_{6,n} = \tfrac{1}{2}\int_{\Sigma\times]0,1[} \vec{F}\left(s,\tau,\tilde{\omega},\tilde{u}^B\right)\cdot\vec{n}\,\theta\varphi d\beta d\mathcal{H}^p_{(s,\tau)},$$

where $\tilde{\omega}$ represents the "trace process" related to ω. Therefore we have

$$\Big|\lim_{n\to+\infty} I'_{6,n} - \tfrac{1}{2}\int_{\Sigma\times]0,1[} \vec{F}\left(s,\tau,\tilde{\omega},u^B\right)\cdot\vec{n}\theta\varphi d\beta d\mathcal{H}^p_{(s,\tau)}\Big| \leq C^{st}\,\delta,$$

where δ is any positive real number; hence, the limit of the first integral in $I_{6,n}$. The same reasoning for the second integral of $I_{6,n}$ then leads to

$$\lim_{n\to+\infty} I_{6,n} = \tfrac{1}{2}\int_{\Sigma\times]0,1[}\Big(\vec{F}\left(t,\sigma,\tilde{\omega},u^B\right)+\vec{F}\left(t,\sigma,\tilde{\pi},\hat{u}^B\right)\Big).\vec{n}\,\theta\varphi d\alpha d\mathcal{H}^p. \tag{3.8}$$

Consequently, adding up the limits (3.2) – (3.8) results in

$$\int_{Q\times]0,1[^2}\Big(\partial_t(\theta\varphi)\,\Delta(id_{\mathbb{R}}) + \vec{B}\cdot\vec{\nabla}(\theta\,\varphi)\,\Delta(f) - \theta\varphi\Delta(g)\Big)\,d\alpha d\beta dx dt$$
$$\geq \tfrac{1}{2}\int_{\Sigma\times]0,1[}\vec{\mathbb{D}}_{(t,\sigma)}\left(\tilde{\pi},\tilde{\omega},u^B,\hat{u}^B\right)\cdot\vec{n}\,\theta\varphi d\alpha d\mathcal{H}^p$$

for all functions φ of $\mathcal{D}_+\left(]0,T[\times\bar{\Omega}\right)$, where

$$\vec{\mathbb{D}}_{(a,b)}\left(\phi,\psi,u^B,\hat{u}^B\right) = \vec{F}\left(a,b,\phi,\hat{u}^B\right) - \vec{F}\left(a,b,\phi,u^B\right) + \vec{F}\left(a,b,\psi,u^B\right) - \vec{F}\left(a,b,\psi,\hat{u}^B\right).$$

By density this inequality is still fulfilled for all functions φ such that $\theta\varphi$ is a positive element of $W^{1,1}(Q)$, with $\theta(0,\cdot)\varphi(0,\cdot) = \theta(T,\cdot)\varphi(T,\cdot) = 0$. So, with a view to establishing the Kruskov relation (3.1), one considers the following test-function in the previous inequality

$$\varphi(t,x) = \begin{cases} \frac{1}{\theta(t,x)}\xi(t), & \text{if } \theta(t,x) > \delta, \\ \frac{1}{\delta}\xi(t), & \text{else}, \end{cases}$$

ξ being any element of $\mathcal{D}_+(0,T)$ and δ being a strictly positive real number ($\delta < \|\theta\|_\infty$). The next inequality follows

$$B_{1,\delta} + B_{2,\delta} \geq 0,$$

where, by denoting $[0<\theta\leq\delta] = \{(t,x)\in Q,\ 0<\theta(t,x)\leq\delta\}$,

$$B_{2,\delta} = \frac{1}{\delta}\int_{[0<\theta\leq\delta]\times]0,1[^2}\Big(\partial_t(\theta\xi)\,\Delta(id_{\mathbb{R}}) + \xi\vec{B}\cdot\vec{\nabla}\theta\,\Delta(f) - \theta\xi\Delta(g)\Big)\,d\alpha d\beta dx dt$$
$$+\frac{1}{2\delta}\int_{\Sigma\cap[0<\theta\leq\delta]\times]0,1[}\vec{\mathbb{D}}_{(t,\sigma)}\left(\tilde{\pi},\tilde{\omega},u^B,\hat{u}^B\right)\cdot\vec{n}\theta\xi\,d\alpha d\mathcal{H}^p.$$

So as to calculate the limit of $B_{2,\delta}$ when δ goes to 0^+, the following estimates are used

$$\frac{1}{\delta}|\pi-\omega| \leq 1 \text{ a.e. on } [\theta\leq\delta],\ \text{ and } \frac{1}{\delta}\left|u^B-\hat{u}^B\right| \leq 1 \text{ a.e. on } \Sigma\cap[0<\theta\leq\delta]$$

and for all real numbers a, b, c, d,

$$|\vec{\mathbb{D}}_{(a,b)}\left(c,d,u^B,\hat{u}^B\right)| \leq 2M'_f\|\vec{B}\|_\infty\left|u^B-\hat{u}^B\right|. \tag{3.9}$$

Hence, there exists a constant C^{st}, such that

$$|B_{2,\delta}| \leq C^{st}\Big(\int_{[0<\theta\leq\delta]}\{\delta + |\partial_t\theta| + \left|\vec{\nabla}\theta\right|\}\,dx dt + d\mathcal{H}^p\left(\Sigma\cap[0<\theta\leq\delta]\right)\Big).$$

That way $B_{2,\delta}$ goes to 0 with δ and we use the Lebesgue dominated convergence theorem to obtain the limit of $B_{1,\delta}$.

Lastly, by resuming a writing through the EMVS ν, ϖ corresponding to the processes π and ω, and using the Lipschitz property of g, the inequality (3.9) gives us, for all functions ξ of $\mathcal{D}_+ (0,T)$,

$$\begin{aligned} &-\int_Q \xi' \int_{\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\; dxdt \\ &\le M_g' \int_Q \xi \int_{\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\; dxdt + M_f' \|\vec{B}\|_\infty \int_\Sigma \left|u^B - \hat{u}^B\right| \xi\, d\mathcal{H}^p. \end{aligned}$$

Hence the time function $h(\cdot)$ defined through

$$\begin{aligned} h : t \to & \int_{\Omega\times\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\; dx \\ & - M_g' \int_0^t \int_{\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(s,x)}(\lambda)\, d\varpi_{(s,x)}(\kappa)\; dxds \\ & - M_f' \|\vec{B}\|_\infty \int_0^t \int_\Gamma \left|u^B - \hat{u}^B\right| \xi\, d\mathcal{H}^{p-1} ds \end{aligned}$$

is non increasing on $[0,T]$. Since it is also a bounded function, it has bounded variations on $[0,T]$. So this function has a trace at $t=0^+$ which is the trace of the BV-function

$$t \to \int_{\Omega\times\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\; dx \quad \text{at} \quad t=0^+.$$

The decomposition

$$\begin{aligned} &\int_{\Omega\times\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\; dx \\ &\le \int_\Omega |u_0-\hat{u}_0|\, dx + \int_{\Omega\times\mathbb{R}^2} |\kappa-\hat{u}_0|\, d\varpi_{(t,x)}(\kappa)\; dx + \int_{\Omega\times\mathbb{R}^2} |\lambda-u_0|\, d\nu_{(t,x)}(\lambda)\; dx \end{aligned}$$

and the definition (1.9) of the initial condition for ν and ϖ give, for a.e. t of $]0,T[$,

$$h(t) \le \int_\Omega |u_0-\hat{u}_0|\, dx.$$

Then, the Gronwall lemma leads to the desired relation (3.1), which completes the proof of Theorem 3.1.

3.2. Conclusion

According to the relation (3.1) established in Theorem 3.1, if ν and ϖ are two EMVS to the problem $(\mathcal{P})$ related to the same boundary conditions, then

$$\int_{\Omega\times\mathbb{R}^2} |\lambda-\kappa|\, d\nu_{(t,x)}(\lambda)\, d\varpi_{(t,x)}(\kappa)\, dx = 0 \quad \text{for a.e.} \quad t \quad \text{in} \quad]0,T[.$$

Thus, the two probability measures $d\nu_{(t,x)}$ and $d\varpi_{(t,x)}$ are equal to a Dirac measure centered on a point noted $u(t,x)$. Moreover, u is a measurable and bounded function on Q, since the constraint (1.6) on the supports of $d\nu_{(t,x)}$ and $d\varpi_{(t,x)}$ is translated into : $0 \le u(t,x) \le \theta(t,x)$, for a.e. (t,x) on Q.

Hence, referring to the equivalent definition of an EMVS to $(\mathcal{P})$ given in Theorem 1.1, we may say that

Theorem 3.2. *The bilateral obstacle problem* $(\mathcal{P})$ *has a unique Entropy Solution* u *which is characterized by the formulation*

$$0 \leq u(t,x) \leq \theta(t,x) \text{ for a.e. } (t,x) \text{ in } Q,$$

and for all real numbers k *of* $[0,1]$, *for all positive functions* ξ *of* $H^1(Q)$, *for all Young measure traces* μ *on* Σ *related to* δ_u, *the next relation holds :*

$$\int_Q \mathcal{L}(t,x,u,k\theta,\xi)\,dxdt + \int_\Omega |u_0 - k\theta(0,\cdot)|\,\xi(0,\cdot)\,dx$$
$$\geq - \int_{\Sigma\times\mathbb{R}} \vec{F}\left(t,\sigma,\lambda,u^B\right)\cdot\vec{n}\,\xi\,d\mu(t,\sigma,\lambda) + \int_\Sigma \vec{F}\left(t,\sigma,k\theta,u^B\right)\cdot\vec{n}\,\xi\,d\mathcal{H}^p.$$

Furthermore, if u *and* v *are two entropy solutions for* $(\mathcal{P})$ *corresponding respectively to the couples of boundary conditions* $\left(u^B,u_0\right)$ *and* $\left(v^B,v_0\right)$, *then for a.e.* t *of* $]0,T[$,

$$\int_\Omega |u-v|\,dx \leq \Big(2M_f'\|\vec{B}\|_\infty \int_0^t \int_\Gamma |u^B - v^B| d\mathcal{H}^{p-1} ds + \int_\Omega |u_0 - v_0|\,dx\Big) e^{M_g' t}.$$

To conclude, let us remember that, according to the works of R.J.Diperna[4], u is the limit in $L^q(Q)$, $1 \leq q < +\infty$, and a.e. on Q, of the whole sequence of penalized solutions $(u_\eta)_{\eta>0}$, when η goes to 0^+. The latter property is used in [levi3] to give some behavior and sensitivity properties of u with respect to θ, in the case of the Cauchy problem in $\mathbb{R}^p$.

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