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Lewy–Stampacchia's inequality for a stochastic T-monotone obstacle problem

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Abstract
The aim of this work is to study a stochastic obstacle problem governed by a T-monotone operator, a random force and a multiplicative stochastic reaction in the frame of Sobolev spaces. After proving a result of existence and uniqueness of the variational solution, by using an *ad hoc* perturbation of the stochastic reaction and a penalization of the constraint, we prove Lewy–Stampacchia's inequalities associated with the problem finally.

Keywords Stochastic PDEs · Obstacle problem · Wiener process · Lewy–Stampacchia's inequality

Mathematics Subject Classification 35K86 · 35R35 · 60H15 · 47J20

1 Introduction

In this paper, we are interested in proving the existence and uniqueness of a solution u to some obstacle problems which can be written (formally)

$$\partial I_K(u) \ni f - \partial_t\Big(u - \int_0^{\cdot} G(u,\cdot)dW\Big) - A(u,\cdot),$$

where K is a closed convex subset of $L^p(\Omega_T, V)$ related to the stochastic constraint ψ, A is a nonlinear T-monotone operator defined on a space V, $(\Omega, (\mathcal{F}_t)_{t\geq 0}, P)$

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is a filtered probability space with the usual assumptions and $W(t)$ is a Wiener process in some separable Hilbert space H. Then, we give the corresponding Lewy–Stampacchia's (L–S) inequalities, namely

$$0 \leq \partial_t\Big(u - \int_0^{\cdot} G(u, \cdot)dW\Big) + A(u, \cdot) - f \leq \Big(f - \partial_t\Big(\psi - \int_0^{\cdot} G(\psi, \cdot)dW\Big) - A(\psi, \cdot)\Big)^- .$$

Concerning stochastic obstacle problems, without seeking to be exhaustive, let us mention the papers of Haussmann and Pardoux [14] where the authors proved the well-posedness of the reflected parabolic problem governed by a bounded linear operator. The question of the semi-linear case was studied by Rascanu [21], Donati-Martin and Pardoux [10], and Xua and Zhang [28]. A penalty method approach is used as in the deterministic case. We also cite the recent book of Zambotti [29] where a study of the nonlinear heat equation with an additive noise is considered.

Several studies on the quasilinear case have been proposed by Denis, Matoussi and Zhang. In [6], a homogeneous SPDE with obstacle, under Lipschitz hypotheses and L^2-integrability conditions on the coefficients, have been studied by using technics of Parabolic potential theory. After the introduction of the notion of parabolic capacity, the authors constructed a solution which admits a quasi-continuous version *via* the penalization method by mixing pathwise arguments and some existence result of the deterministic obstacle problem. The result has been extended in [7] by considering a weaker $L^{p,q}$-integrability conditions on the coefficients. Then, they used the same approach to study the case of non-homogeneous operator as they derived also a local maximum principle in [8].

In a differential inclusion approach, we mention the works of Rascanu [22] and Bensoussan and Rascanu [4] where a maximal monotone operator is considered on a Hilbert space; Barbu [2] for nonlinear heat problems and Bauzet et al. [3] for an Allen–Cahn type equation.

Concerning monotone operators in a non-Hilbertian case, Rascanu and Rotenstein [23] were interested, among other things, in strong solutions to some stochastic variational inequalities when the barriers cancel the diffusion coefficients. Our aim in this paper is to revisit similar variational inequalities by adding random dependences for the operator, the source and the stochastic reaction terms, and the obstacle. We will in particular assume that $f - \partial_t\Big(\psi - \int_0^{\cdot} G(\psi, \cdot)dW\Big) - A(\psi, \cdot)$ can be written as the difference of two non-negative elements of a dual-space to derive Lewy–Stampacchia's inequalities. Then, we propose in the appendix-section some extensions to situations where the obstacle and the solution are not in the same space, or to bilateral obstacle problems.

There exists a vast literature on deterministic L–S inequalities. Lewy and Stampacchia [16] proved the first inequalities in the context of super-harmonic problems. Then, many authors have been interested in the so-called Lewy–Stampacchia's inequality associated with obstacle problems. Let us cite the monograph of Rodrigues [24] for hyperbolic problems and the paper of Mokrane et al. [18] for elliptic problems in the context of Sobolev spaces with variable exponents. Concerning parabolic problems, the first result is the one of Donati [9] for problems with a monotone operator. Recently, Guibé et al. [13] extended this result to the frame of Leray-Lions pseudomonotone operators.

To the best of the author's knowledge, there doesn't exist in the literature a result of existence and uniqueness associated with corresponding L–S inequalities, of the solution to a stochastic obstacle problem with a nonlinear operator associated with a random obstacle that doesn't cancel the diffusion coefficients. Our aim, in this paper, is to propose such a result with general assumptions on the T-monotone operator and a general multiplicative noise. By using a penalization method of the constraint, associated with a suitable perturbation of the stochastic reaction to formally lead to an additive stochastic source on the free-set where the constraint is violated, we are able to prove on one hand the existence of a solution to the stochastic obstacle problem, and on the other hand, to prove the corresponding stochastic Lewy–Stampacchia's inequalities.

The paper is organized in the following way: after giving the hypotheses, a result of uniqueness (Lemma 1) and the main result (Theorem 1) in Sects. 2, 3 is devoted to the proof of the results. A first step concerns the existence of a solution to the approximating problem associated with a parameter ϵ. Additionally, some *a priori* estimates and passage to the limit with respect to ϵ are considered when h^- is a regular non-negative element. A first proof of Lewy–Stampacchia's inequality is given when h^- is still regular. Finally, the proof of Lewy–Stampacchia's inequality is extended to the general case. In a next small section we present some numerical illustrations, then we finish with an appendix where some possible extensions are presented concerning obstacles with negative values on the boundary and the case of bilateral obstacles.

2 Notation and hypotheses

Let us denote by $D \subset \mathbb{R}^d$ a Lipschitz bounded domain, $T > 0$ and by $p \in (1, +\infty)$. As usual, p' is the conjugate exponent of p, $V = W_0^{1,p}(D)$ if $p \geq 2$ and $V = W_0^{1,p}(D) \cap L^2(D)$ with the graph-norm else. $W_0^{1,p}(D)$ denotes the sub-space of elements of $W^{1,p}(D)$ with null trace, endowed with Poincaré's norm, and $H = L^2(D)$ is identified with its dual space so that, the corresponding dual spaces to V are $V' = W^{-1,p'}(D)$ if $p \geq 2$ and $V' = W^{-1,p'}(D) + L^2(D)$ else (cf. e.g. [11, p.24]). The duality bracket for $T \in V'$ and $v \in V$ is denoted $\langle T, v\rangle$.
The presentation of our results is in an abstract way so that one can easily extend them to more general Riesz separable reflexive Banach spaces V. We will not develop this point of view because V will not be a Banach lattice.

In our situation, the Lions–Guelfand triple $V \hookrightarrow_d H = L^2(D) \hookrightarrow_d V'$ holds.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space (*e.g.* the classical Wiener space) endowed with a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ completed with respect to the measure P. $W(t)$ is an adapted Wiener process in H with nuclear covariance operator Q with $tr\,Q < \infty$. Denote by $\Omega_T = (0, T) \times \Omega$ and $\mathcal{P}_T$ the predictable σ-algebra on Ω_T.[1]

Let $L_Q(H)$ denotes the spaces of linear operators Φ defined on $Q^{\frac{1}{2}}H$ with values in H such that $\Phi Q^{\frac{1}{2}} \in \mathcal{HS}(H)$ (the space of Hilbert-Schmidt operators from H to

[1] $\mathcal{P}_T := \sigma(\{]s, t] \times F_s | 0 \leq s < t \leq T, F_s \in \mathcal{F}_s\} \cup \{\{0\} \times F_0 | F_0 \in \mathcal{F}_0\})$ (see [17, p. 33]). Then, a process defined on Ω_T with values in a given space E is predictable if it is $\mathcal{P}_T$-measurable.

H). $L_Q(H)$ is a separable Hilbert space relatively to the scalar product $(\Phi, \Psi)_Q = tr \Phi Q^{\frac{1}{2}}(\Psi Q^{\frac{1}{2}})^*$. The norm in this space is denoted by $|\cdot|_Q$. We recall that the stochastic integrals over a Wiener process are defined for predictable operators B such that $E[\int_0^t |B(s)|_Q^2 ds] < \infty$ for any $t \geq 0$ [15, Sect. I-2].

We will consider in the sequel the following assumptions:

H_1 : Let $A : V \times \Omega_T \to V'$, $G : H \times \Omega_T \to L_Q(H)$, $\psi : \Omega_T \to V$, $f : \Omega_T \to V'$ and $u_0 : \Omega \to H$ such that:

$H_{1,1}$: For any $v \in V$ and $u \in H$, $A(v, \cdot)$, $G(u, \cdot)$, ψ and f are predictable.
$H_{1,2}$: u_0 is $\mathcal{F}_0$-measurable.

H_2 : $\exists \alpha, \bar{K} > 0, \lambda_T, \lambda \in \mathbb{R}$, $l_1 \in L^1(\Omega_T)$ and $g \in L^{p'}(\Omega_T)$, both predictable, such that:

$H_{2,1}$: $(t,\omega) \in \Omega_T$ a.e., $\forall v \in V$, $\langle A(v,t,\omega), v\rangle + \lambda\|v\|_H^2 + l_1(t,\omega) \geq \alpha\|v\|_V^p$.
$H_{2,2}$: ($T-$ monotonicity [19, p. 120]) $(t,\omega) \in \Omega_T$ a.e., $\forall v_1, v_2 \in V$,

$$\lambda_T(v_1 - v_2, (v_1 - v_2)^+)_H + \langle A(v_1,t,\omega) - A(v_2,t,\omega), (v_1 - v_2)^+\rangle \geq 0.$$

Note that since $v_1 - v_2 = (v_1 - v_2)^+ - (v_2 - v_1)^+$, $\lambda_T Id + A$ is also monotone.
$H_{2,3}$: $(t,\omega) \in \Omega_T$ a.e., $\forall v \in V$, $\|A(v,t,\omega)\|_{V'} \leq \bar{K}\|v\|_V^{p-1} + g(t,\omega)$.
$H_{2,4}$: (Hemi-continuity) $(t,\omega) \in \Omega_T$ a.e., $\forall v, v_1, v_2 \in V$,
$\eta \in \mathbb{R} \mapsto \langle A(v_1 + \eta v_2, t, \omega), v\rangle$ is continuous.

Remark 1 Assumptions $H_{2,2}$ and $H_{2,4}$ yield (e.g. [25, Lemma 2.16 p.38]) the continuity of $\lambda_T Id + A$, thus of A with respect to v from V-strong to V'-weak. Thus, for any $v_1 \in V$, the application $A_{v_1} : V \times \Omega_T \to \mathbb{R}, (v,t,\omega) \mapsto \langle A(v,t,\omega), v_1\rangle$ is a Carathéodory function.
Therefore, it is $\mathcal{B}(V)\otimes\mathcal{P}_T$ measurable and, $\langle A(v(t,\omega),t,\omega), v_1\rangle$ is predictable too for any $V-$valued predictable process v [5, p.9]. If V is separable, $A(v,\cdot)$ is predictable with values in $(V', \mathcal{B}(V'))$ since the weak and the strong measurabilities coincide thanks to Pettis's Theorem.

H_3 : $\exists M > 0$ and $l \in L^1(\Omega_T)$, predictable, such that

$H_{3,1}$: $(t,\omega) \in \Omega_T$ a.e., $\forall \theta, \sigma \in H$, $|G(\theta,t,\omega) - G(\sigma,t,\omega)|_Q^2 \leq M\|\theta - \sigma\|_H^2$.
$H_{3,2}$: $(t,\omega) \in \Omega_T$ a.e., $\forall u \in H$, $|G(u,t,\omega)|_Q^2 \leq l(t,\omega) + M\|u\|_H^2$.

Remark 2 Thanks to Assumption H_3, $G : L^2(D) \times \Omega_T \to L_Q(L^2(D))$ is a Carathéodory function. It is $\mathcal{B}(H) \otimes \mathcal{P}_T$ measurable and, $G(u(t,\omega),t,\omega)$ is predictable too for any H-valued predictable process u.

H_4 : $\psi \in L^p(\Omega, L^p(0,T,V))$, $\partial_t\Big(\psi - \int_0^\cdot G(\psi,\cdot)dW\Big) \in L^{p'}(\Omega_T, V')$ predictables.

Definition 1 Denote by K the convex set of admissible functions

$$K = \{v \in L^p(\Omega_T, V), \quad v(x,t,\omega) \geq \psi(x,t,\omega) \quad \text{a.e. in } D \times \Omega_T\}.$$

H_5 : $f \in L^{p'}(\Omega_T, V')$ is predictable and one assumes moreover that

$$h = f - \partial_t\Big(\psi - \int_0^\cdot G(\psi,\cdot)dW\Big) - A(\psi,\cdot) \in L^p(\Omega_T, V)^*,$$
$$\text{where}\quad L^p(\Omega_T, V)^* = (L^{p'}(\Omega_T, V'))^+ - (L^{p'}(\Omega_T, V'))^+ \subset L^{p'}(\Omega_T, V')$$

denotes the order dual: the difference of two non-negative elements of $L^{p'}(\Omega_T, V')$, i.e. $h = h^+ - h^-$ where $h^+, h^- \in (L^{p'}(\Omega_T, V'))^+$ are non-negative elements of $L^{p'}(\Omega_T, V')$. f, h^+, h^- are also assumed to be predictable.
We recall that $h^\pm \in (L^{p'}(\Omega_T, V'))^+$ in the sense:

$$\forall \varphi \in L^p(\Omega_T, V), \quad \varphi \geq 0 \Rightarrow E\int_0^T \langle h^\pm, \varphi\rangle ds \geq 0.$$

H_6 : $u_0 \in L^2(\Omega, H)$ satisfies the constraint, *i.e.* $u_0 \geq \psi(0)$.

Our aim is to look for (u, k), in a space defined straight after, solution to

$$\begin{cases} du + A(u,\cdot)ds + kds = fds + G(u,\cdot)dW & \text{in} \quad D\times\Omega_T, \\ u(t=0) = u_0 & \text{in} \quad H,\ a.s., \\ u \geq \psi & \text{in} \quad D\times\Omega_T, \\ u = 0 & \text{on} \quad \partial D\times\Omega_T, \\ \langle k, u-\psi\rangle = 0 \ \text{ and } \ k \leq 0 & \text{in} \quad \Omega_T. \end{cases} \tag{1}$$

Remark 3 Taking into account Assumptions H_4 and H_5, it's worth noticing that ψ solves the following stochastic problem

$$d\psi + A(\psi,\cdot)dt = G(\psi,\cdot)dW + (f-h)dt,$$

and the obstacle can be understood as a constraint in the coupling of stochastic PDEs. For exemple, $\psi(t,x) = sin(\pi x)\exp(\beta(t) - \pi^2 t)$ in $(0,T)\times(0,1)\times\Omega$ where β is the standard Brownian motion. ψ is the solution to $\partial_t(\psi - \int_0^\cdot \psi dW) - \partial_x^2\psi - \frac{1}{2}\psi = 0$ and satisfies H_4 and H_5 with $p=2$, $A(v,\cdot) = \partial_x^2 v - \frac{1}{2}v$ and $G(v,\cdot) = v$.
When the obstacle is with values in V, one can observe that the problem can reduce to the question of a positivity obstacle problem with a stochastic reaction term vanishing at 0.
Indeed, by setting $\hat{u} = u - \psi$, $\hat{u}_0 = u_0 - \psi(0)$, $\hat{A}(\hat{u}) = A(\hat{u}+\psi) - A(\psi)$, with $\hat{G}(\hat{u}) = G(\hat{u}+\psi) - G(\psi)$ and $\hat{f} = f - \partial_t\big(\psi - \int_0^\cdot G(\psi)dW\big) - A(\psi)$, the equation becomes $d\hat{u} + \hat{A}(\hat{u},.)ds + \hat{k}ds = \hat{f}ds + \hat{G}(\hat{u},.)dW$ in $D\times\Omega_T$ with $\hat{G}(0,.) = 0$ and under the constraint $\hat{u} \geq 0$.
In case the obstacle ψ is not with values in V, if for example ψ has non-positive values on the boundary of D, or in case of a bilateral obstacle problem, this change of problem may not be helpful and we present some extensions in this direction in the appendix-section 5.

Let us introduce the concept of a solution for Problem (1).

Definition 2 The pair (u,k) is a solution to Problem (1) if:

- $u\in L^p(\Omega_T,V)$ and $k\in L^{p'}(\Omega_T,V')$ are predictable, $u\in L^2(\Omega,\mathcal{C}([0,T],H))$.
- $u(t=0)=u_0$ and $u\in K$.
- P-a.s, for all $t\in[0,T]$,

$$u(t)+\int_0^t kds+\int_0^t A(u,\cdot)ds=u_0+\int_0^t G(u,\cdot)dW(s)+\int_0^t fds.$$

- $-k\in(L^{p'}(\Omega_T,V'))^+$ and $\forall v\in K, \langle k,u-v\rangle\ge 0$ a.e. in Ω_T.

Remark 4 Since the embedding $V\hookrightarrow H$ is continuous, u is equally a predictable process with values in H or in V (thanks to Kuratowski's theorem [27, Th. 1.1 p. 5]).

Remark 5 We remind that $(L^p(\Omega_T,V))^+ = \{u\in L^p(\Omega_T,V),\ u(t,\omega)\in V^+$ a.e. in $\Omega_T\}$, therefore, $-k\in(L^{p'}(\Omega_T,V'))^+$ if and only if $-k(t,\omega)\in(V')^+$, a.e. in Ω_T.
Indeed, If one assumes first that $-k\in(L^{p'}(\Omega_T,V'))^+$. Then, for any given $\varphi\in V^+$, any $A\in\mathcal{F}$ and any $B\in\mathcal{B}(0,T)$, $1_{A\times B}\varphi\in(L^p(\Omega_T,V))^+$.
Thus, $\int_{A\times B}\langle k(t,\omega),\varphi\rangle dtdP\le 0$ for any such A and B and $\langle k(t,\omega),\varphi\rangle\le 0$ on a subset of Ω_T of full measure, depending *a priori* on φ.
Since V is separable, for a given dense family $\{\varphi_n, n\in\mathbb{N}\}\subset V$, there exists $\widetilde{\Omega}_T\subseteq\Omega_T$ a subset of full measure such that $\langle k(t,\omega),\varphi_n^+\rangle\le 0$ for any n and all $(t,\omega)\in\widetilde{\Omega}_T$.
Let $\varphi\in V^+$ and $(\varphi_l)\subset\{\varphi_n, n\in\mathbb{N}\}$ satisfying $\varphi_l\to\varphi$ in V. Thus, $\varphi_l^+\to\varphi^+=\varphi$ in V and since $\langle k(t,\omega),\varphi_l^+\rangle\le 0$, the same inequality holds for φ. Thus, $-k(t,\omega)\in(V')^+$, $(t,\omega)\in\Omega_T$ a.e.
The converse is immediate: if $\varphi(t,\omega)\in V^+$ a.e. in Ω_T, one gets $\langle k(t,\omega),\varphi(t,\omega)\rangle\le 0$ a.e. in Ω_T and $\int_{\Omega_T}\langle k(t,\omega),\varphi(t,\omega)\rangle dtdP\le 0$.
As a consequence, knowing that $-k\in(L^{p'}(\Omega_T,V'))^+$ and $u\in K$, imply that the condition: $\langle k,u-\psi\rangle=0$ a.e. in Ω_T is equivalent to the condition: $\forall v\in K, \langle k,u-v\rangle\ge 0$ a.e. in Ω_T.

Let us state our main result.

Theorem 1 *Under Assumptions* (H_1)*–*(H_6)*, there exists a unique solution* (u,k) *to Problem* (1) *in the sense of Definition* 2*. Moreover, the following Lewy–Stampacchia's inequality holds*

$$0\le\partial_t\Big(u-\int_0^\cdot G(u,\cdot)dW\Big)+A(u,\cdot)-f\le h^-=\Big(f-\partial_t\Big(\psi-\int_0^\cdot G(\psi,\cdot)dW\Big)-A(\psi,\cdot)\Big)^-.$$

Remark 6 Note that Problem (1) can be written in the equivalent form:

$$\partial I_K(u)\ni f-\partial_t\Big(u-\int_0^\cdot G(u,\cdot)dW\Big)-A(u,\cdot)$$

where $\partial I_K(u)$ represents the sub-differential of $I_K : L^p(\Omega_T, V) \to \bar{\mathbb{R}}$ defined as

$$I_K(u) = \begin{cases} 0, & u \in K, \\ +\infty, & u \notin K, \end{cases}$$

and $\partial I_K(u) = N_K(u) = \{y \in L^{p'}(\Omega_T, V'); E\int_0^T \langle y, u - v\rangle ds \geq 0, \forall v \in K\}$ (see [1, p. 7 − 8]).

Before entering in the proof of our main theorem, we start with the following result.

Lemma 1 *If* (u_1, k_1) *and* (u_2, k_2) *are two solutions to* (1) *in the sense of Definition* 2 *associated with two different forces* f_1 *and* f_2 *then: there exists a positive constant* $C > 0$ *such that*

$$E \sup_{t\in[0,T]} \|(u_1 - u_2)(t)\|_H^2 \leq C\|f_1 - f_2\|_{L^{p'}(\Omega_T,V')}\|u_1 - u_2\|_{L^p(\Omega_T,V)}.$$

Remark 7 Note that Lemma 1 ensures the uniqueness of the solution to (1) in the general framework.

Proof For any $t \in [0, T]$ and P-a.s we have

$$\begin{aligned} &u_1(t) - u_2(t) + \int_0^t [k_1 - k_2]ds + \int_0^t [A(u_1, \cdot) - A(u_2, \cdot)]ds \\ &= \int_0^t [G(u_1, \cdot) - G(u_2, \cdot)]dW(s) + \int_0^t [f_1 - f_2]ds. \end{aligned}$$

Applying Ito's formula with $F(t, v) = \frac{1}{2}\|v\|_H^2$, one gets for any $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\|(u_1 - u_2)(t)\|_H^2 + \int_0^t \langle A(u_1, \cdot) - A(u_2, \cdot), u_1 - u_2\rangle ds + \int_0^t \langle k_1 - k_2, u_1 - u_2\rangle ds \\ &= \int_0^t \langle f_1 - f_2, u_1 - u_2\rangle ds + \int_0^t \langle [G(u_1, \cdot) - G(u_2, \cdot)]dW(s), u_1 - u_2\rangle \\ &+ \frac{1}{2}\int_0^t |G(u_1, \cdot) - G(u_2, \cdot)|_Q^2 ds. \end{aligned}$$

- Since $u_1, u_2 \in K$, Remark 5 yields a.e. in Ω_T,

  $$\langle k_1 - k_2, u_1 - u_2\rangle = \langle k_1, u_1 - u_2\rangle + \langle k_2, u_2 - u_1\rangle \geq 0.$$

  Therefore, for any t

  $$\int_0^t \langle k_1 - k_2, u_1 - u_2\rangle ds = \int_0^t \langle k_1, u_1 - u_2\rangle + \langle k_2, u_2 - u_1\rangle ds \geq 0 \ a.s.$$

- $\forall t \in [0,T]$, $\quad \frac{1}{2}\int_0^t |G(u_1,\cdot)-G(u_2,\cdot)|_Q^2 ds \leq M\int_0^t \|u_1-u_2\|_H^2 ds.$
- Since $\lambda_T Id + A$ is T-monotone, $\forall t \in [0,T]$,

$$\int_0^t \langle A(u_1,\cdot)-A(u_2,\cdot), u_1-u_2\rangle ds \geq -\lambda_T \int_0^t \|u_1-u_2\|_H^2 ds.$$

- By Burkhölder–Davis–Gundy's inequality [15, Theorem 2.5 p.1240] (see also [23, p.652]) and Young's inequality, there exists a positive δ such that

$$\begin{aligned} &E\left[\sup_{t\in[0,T]} \left|\int_0^t \langle [G(u_1,\cdot)-G(u_2,\cdot)]dW(s), u_1-u_2\rangle\right|\right] \\ &\quad \leq \frac{3\delta}{2} E\sup_{t\in[0,T]} \|(u_1-u_2)(t)\|_H^2 + \frac{3M}{2\delta} E\int_0^T \|(u_1-u_2)(s)\|_H^2 ds. \end{aligned}$$

- $E\sup_{t\in[0,T]} \left|\int_0^t \langle f_1-f_2, u_1-u_2\rangle ds\right| \leq \|f_1-f_2\|_{L^{p'}(\Omega_T,V')}\|u_1-u_2\|_{L^p(\Omega_T,V)}.$

With a convenient choice of δ (*e.g.* $\delta=\frac{1}{4}$), we deduce the existence of a positive constant c such that

$$\begin{aligned} E\sup_{t\in[0,T]} \|(u_1-u_2)(t)\|_H^2 \leq & c\|f_1-f_2\|_{L^{p'}(\Omega_T,V')}\|u_1-u_2\|_{L^p(\Omega_T,V)} \\ & + c\int_0^T E\sup_{\tau\in[0,s]} \|(u_1-u_2)(\tau)\|_H^2 ds. \end{aligned} \tag{2}$$

Then, Gronwall's lemma ensures that

$$E\sup_{t\in[0,T]} \|(u_1-u_2)(t)\|_H^2 \leq ce^{cT}\|f_1-f_2\|_{L^{p'}(\Omega_T,V')}\|u_1-u_2\|_{L^p(\Omega_T,V)}. \tag{3}$$

□

3 Proof of Theorem 1

We will prove Theorem 1 in three steps:

- Existence of the solution and a first Lewy–Stampacchia's inequality, assuming that h^- is regular.
- A second Lewy–Stampacchia's inequality, still with a regular h^-.
- Proof of the main theorem in the general case.

3.1 Existence of the solution and a first Lewy–Stampacchia's inequality, assuming that h^- is regular.

3.1.1 Penalization

Let $\epsilon > 0$ and consider the following approximating problem:

$$\begin{cases} u_\epsilon(t) + \displaystyle\int_0^t (A(u_\epsilon, \cdot) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - f)ds = u_0 + \int_0^t \widetilde{G}(u_\epsilon, \cdot)dW(s) \\ u_\epsilon(0) = u_0, \end{cases} \tag{4}$$

where $\tilde{q} = \min(p, 2)$ and $\widetilde{G}(u_\epsilon, \cdot) = G(\max(u_\epsilon, \psi), \cdot)$. The idea of the perturbation of G is to have formally an additive stochastic source on the free-set where the constraint is violated.

Note that $\widetilde{G}$ satisfies also Assumptions H_1 and H_3, as well as Assumption H_5 since $\widetilde{G}(\psi, \cdot) = G(\psi, \cdot)$. Indeed, since ψ is predictable in H, $max(u, \psi)$ is also predictable for any $u \in H$ and $\widetilde{G}(u, \cdot)$ is predictable thanks to Remark 2.

For any $u, v \in V$, $|\widetilde{G}(u, \cdot) - \widetilde{G}(v, \cdot)|_Q^2 \le M\| \max(u, \psi) - \max(v, \psi)\|_H^2 \le M\|u - v\|_H^2$.

The only difference in the assumptions lies in $H_{3,2}$ where one gets now that

$$|\widetilde{G}(u, \cdot)|_Q^2 \le l + 2M\|\psi\|_H^2 + 2M\|u\|_H^2 = \tilde{l} + \tilde{M}\|u\|_H^2$$

where $\tilde{l}$ is a $L^1(\Omega_T)$-predictable element by composition of functions, depending only on the data.

Consider $\bar{A}(u_\epsilon, \cdot) = A(u_\epsilon, \cdot) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - f$ and note that:

- By construction, $\bar{A}$ is an operator defined on $V \times \Omega_T$ with values in V'.
- Since ψ and f are predictable with values in V and V' respectively, $u \mapsto u^-$ is a Lipschitz-continuous mapping then, for any $v \in V$, $-\frac{1}{\epsilon}[(v - \psi)^-]^{\tilde{q}-1} - f$ is predictable with values in V' and therefore $\bar{A}$ satisfies Assumption $H_{1,1}$.
- Since $x \mapsto -x^-$ is non-decreasing, $\lambda_T Id + \bar{A}$ is T-monotone.
- The structure of the penalization operator yields the hemi-continuity of $\bar{A}$ in the sense of $H_{2,4}$.
- (Coercivity): Note that for any $\delta > 0$, there exists $C_{\delta,\epsilon} > 0$ such that: $\forall v \in V$,

$$\begin{aligned} \langle f, v\rangle \le& C_\delta\|f\|_{V'}^{p'} + \delta\|v\|_V^p; \\ \left\langle -\frac{1}{\epsilon}[(v - \psi)^-]^{\tilde{q}-1}, v\right\rangle \ge& \left\langle -\frac{1}{\epsilon}[(v - \psi)^-]^{\tilde{q}-1}, \psi\right\rangle \\ \ge& -\delta\|v\|_{L^{\tilde{q}}(D)}^{\tilde{q}} - C_{\delta,\epsilon}\|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}} \\ \ge& -\delta C\|v\|_V^p - C_{\delta,\epsilon}\|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}} - C_p \end{aligned}$$

where C is related to the continuous embedding of V in $L^{\tilde{q}}(D)$.
Denote by $\tilde{l}_1 = l_1 + C_\delta \|f\|_{V'}^{p'} + C_{\delta,\epsilon}\|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}}$. It is a $L^1(\Omega_T)$ predictable element thanks to the assumptions on f and ψ, depending only on the data. Therefore, by a convenient choice of δ, $\bar{A}$ satisfies $H_{2,1}$ by considering $\tilde{l}_1$ instead of l_1.

- (Boundedness): $\forall v \in V$,

$$\left\| -\frac{1}{\epsilon}[(v-\psi)^-]^{\tilde{q}-1} \right\|_{L^{\tilde{q}'}(D)} = \frac{1}{\epsilon}\|(v-\psi)^-\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} \le C_\epsilon \left(\|v\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} + \|\psi\|_{L^{\tilde{q}}(D)}^{\tilde{q}-1} \right) \le C_\epsilon \left(\|v\|_{L^{\tilde{q}}(D)}^{p-1} + \|\psi\|_{L^{\tilde{q}}(D)}^{p-1} \right) + C_p$$

since $\tilde{q} < p$ may be possible. Now, since the embeddings of $L^{\tilde{q}'}(D)$ in V' and of V in $L^{\tilde{q}}(D)$ are continuous,

$$\left\| -\frac{1}{\epsilon}[(v-\psi)^-]^{\tilde{q}-1} \right\|_{V'} \le C_\epsilon \left(\|v\|_V^{p-1} + \|\psi\|_V^{p-1} \right) + C_p$$

and Assumption $\mathrm{H}_{2,3}$ is satisfied with $\bar{K}$ replaced by $\bar{K} + C_\epsilon$ and g by $\tilde{g} = g + C_\epsilon\|\psi\|_V^{p-1} + f + C_p$ which is a predictable element of $L^{p'}(\Omega_T)$.

- (The noise): Let us denote by $U = Q^{\frac{1}{2}}(H)$, we recall that U is a separable Hilbert space endowed with the scalar product $(u,v)_U = (Q^{-\frac{1}{2}}u, Q^{-\frac{1}{2}}u)_H$ (see [17, Prop. C.0.3 p. 221]) and note that $\widetilde{G} \in HS(Q^{\frac{1}{2}}(H), H)$. Since $(W(t))_{t\in[0,T]}$ is a Wiener process in H with a nuclear covariance operator Q, $(W(t))_{t\in[0,T]}$ is a Cylindrical Wiener process with a covariance operator I in U.

By [17, Th. 4.2.4 p.91] and Remark 4, for all $\epsilon > 0$, there exists a unique solution $u_\epsilon \in L^p(\Omega_T, V)$ predictable such that $u_\epsilon \in L^2(\Omega, \mathcal{C}([0,T], H))$ and satisfying (4) for all $t \in [0,T]$ and P-a.s. in Ω.
Moreover, thanks to [17, Th. 4.2.5 p.91], $(u_\epsilon)_{\epsilon>0}$ is bounded in $L^p(\Omega_T, V) \cap L^2(\Omega_T, H)$.
Thanks to Assumptions $H_{2,3}$, we get the following lemma.

Lemma 2
- *$(u_\epsilon)_{\epsilon>0}$ is bounded in $L^p(\Omega_T, V) \cap C([0,T], L^2(\Omega, H))$.*
- *$(A(u_\epsilon, \cdot))_{\epsilon>0}$ is bounded in $L^{p'}(\Omega_T, V')$.*

Proof Let $\epsilon > 0$ and $v^* \in K$ such that $\partial_t\big(v^* - \int_0^\cdot \widetilde{G}(v^*,\cdot)dW\big) \in L^{p'}(\Omega_T, V')$ with predictable assumptions. Note that $v^* = \psi$ holds in this situation and

$$u_\epsilon(t) - v^*(t) + \int_0^t \Big(A(u_\epsilon,\cdot) - \frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde{q}-1}\Big)ds$$
$$= u_0 - v^*(0) + \int_0^t [f - \partial_t(v^* - \int_0^\cdot \widetilde{G}(v^*,\cdot)dW)]ds$$
$$+ \int_0^t [\widetilde{G}(u_\epsilon,\cdot) - \widetilde{G}(v^*,\cdot)]dW(s).$$

Itô's stochastic energy yields

$$\begin{aligned}
&\|u_\epsilon - v^*\|_H^2(t) + 2\int_0^t \langle A(u_\epsilon,\cdot), u_\epsilon - v^*\rangle ds \\
&\quad - 2\int_0^t \int_D \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1}(u_\epsilon - v^*)dxds - \|u_0 - v^*(0)\|_H^2 \\
&= 2\int_0^t \langle f - \partial_s(v^* - \int_0^\cdot \widetilde{G}(v^*,\cdot)dW), u_\epsilon - v^*\rangle ds \\
&\quad + 2\int_0^t \Big(u_\epsilon - v^*, [\widetilde{G}(u_\epsilon,\cdot) - \widetilde{G}(v^*,\cdot)]dW(s)\Big)_H + \int_0^t |\widetilde{G}(u_\epsilon,\cdot) - \widetilde{G}(v^*,\cdot)|_Q^2 ds.
\end{aligned}$$

Note that

$$\begin{aligned}
&- 2\int_0^t \int_D \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1}(u_\epsilon - v^*)dxds \\
&\quad = -\frac{2}{\epsilon}\int_0^t\int_D [(u_\epsilon - \psi)^-]^{\tilde{q}-1}(u_\epsilon - \psi)dxds \\
&\qquad - \frac{2}{\epsilon}\int_0^t\int_D [(u_\epsilon - \psi)^-]^{\tilde{q}-1}(\psi - v^*)dxds \geq 0, \\
&\langle A(u_\epsilon,\cdot), u_\epsilon - v^*\rangle \geq \alpha\|u_\epsilon\|_V^p - \lambda\|u_\epsilon\|_H^2 - l_1(t,\omega) - \langle A(u_\epsilon,\cdot), v^*\rangle \\
&\quad \geq \alpha\|u_\epsilon\|_V^p - \lambda\|u_\epsilon\|_H^2 - l_1(t,\omega) - \bar{K}\|u_\epsilon\|_V^{p-1}\|v^*\|_V - g(t,\omega)\|v^*\|_V \\
&\quad \geq \frac{\alpha}{2}\|u_\epsilon\|_V^p - \lambda\|u_\epsilon\|_H^2 - l_1(t,\omega) - C(v^*)(t,\omega)
\end{aligned}$$

where $C(v^*) \in L^1(\Omega_T)$. Thus, for any positive γ, Young's inequality yields the existence of a positive constant C_γ that may change form line to line, such that

$$\begin{aligned}
&E\|u_\epsilon - v^*\|_H^2(t) + 2E\int_0^t \frac{\alpha}{2}\|u_\epsilon\|_V^p(s)ds \\
&\quad \leq \lambda E\int_0^t \|u_\epsilon\|_H^2(s)ds + \|l_1 + C(v^*)\|_{L^1(\Omega_T)} + C_\gamma(f, \partial_s(v^* - \int_0^\cdot \widetilde{G}(v^*,\cdot)dW)) \\
&\qquad + \gamma E\int_0^t \|u_\epsilon - v^*\|_V^p(s)ds + E\int_0^t |\widetilde{G}(u_\epsilon,\cdot) - \widetilde{G}(v^*,\cdot)|_Q^2 ds. \\
&\quad \leq CE\int_0^t \|u_\epsilon - v^*\|_H^2(s)ds + \frac{\alpha}{2}E\int_0^t \|u_\epsilon\|_V^p(s)ds + C,
\end{aligned}$$

for a suitable choice of γ and thanks to $H_{3,2}$.
Then, the first part of the lemma is proved by Gronwall's lemma, and the second one by adding $H_{2,3}$ to the first estimate. □

3.1.2 A priori estimates with a regular h^-

H_7: We will assume in this subsection that h^- is a predictable non negative element of $L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$.

Lemma 3 *Under H_7, $(\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1})_{\epsilon>0}$ is bounded in $L^{\tilde{q}'}(\Omega_T \times D)$.*

Proof Let $\delta > 0$ and consider the following approximation from [20, p. 152].

$$F_\delta(r) = \begin{cases} r^2 - \dfrac{\delta^2}{6} & if \ \ r \le -\delta, \\ -\dfrac{r^4}{2\delta^2} - \dfrac{4r^3}{3\delta} & if \ \ -\delta \le r \le 0, \\ 0 & if \ \ r \ge 0. \end{cases} \tag{5}$$

Note that $(-\frac{1}{2}F_\delta')_\delta$ approximates the negative part. Moreover, $F_\delta(\cdot) \in \mathcal{C}^2(\mathbb{R})$, and satisfies:

$$\begin{cases} |F_\delta(r)| \le r^2, \\ |F_\delta'(r)| \le 2r \quad and \quad \forall r \in \mathbb{R},\ F_\delta'(r) \le 0, \\ |F_\delta''(r)| \le \frac{8}{3} \quad and \quad \forall r \in \mathbb{R},\ F_\delta''(r) \ge 0. \end{cases}$$

Set $\varphi_\delta(v) = \int_D F_\delta(v(x))dx, \quad v \in L^2(D)$ and denote by S the set $\{u_\epsilon \le \psi\}$. Applying Ito's formula [20, Th. 4.2 p. 65] (see also [23, Lemma 4]) to the process $u_\epsilon - \psi$, one gets for any $t \in [0, T]$

$$\begin{aligned} &\varphi_\delta(u_\epsilon(t) - \psi(t)) + \int_0^t \langle A(u_\epsilon, \cdot) - A(\psi, \cdot), F_\delta'(u_\epsilon - \psi)\rangle ds \\ &\quad - \frac{1}{\epsilon}\int_0^t \langle [(u_\epsilon - \psi)^-]^{\tilde{q}-1}, F_\delta'(u_\epsilon - \psi)\rangle ds \\ &= \overbrace{\varphi_\delta(u_\epsilon(0) - \psi(0))}^{=0} + \overbrace{\int_0^t \langle h^+, F_\delta'(u_\epsilon - \psi)\rangle ds}^{\le 0} - \int_0^t \langle h^-, F_\delta'(u_\epsilon - \psi)\rangle ds \\ &\quad + \int_0^t (\{\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(\psi, \cdot)\}dW(s), F_\delta'(u_\epsilon - \psi)) \\ &\quad + \frac{1}{2}\int_0^t Tr(F_\delta''(u_\epsilon - \psi)\{\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(\psi, \cdot)\}Q\{\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(\psi, \cdot)\}^*)ds. \end{aligned}$$

Since $\widetilde{G}(u_\epsilon, \cdot) = \widetilde{G}(\psi, \cdot)$ on the set S, we deduce

$$\frac{1}{2}\int_0^t Tr(F_\delta''(u_\epsilon - \psi)\{\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(\psi, \cdot)\}Q\{\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(\psi, \cdot)\}^*)ds = 0.$$

Taking the expectation, one has

$$E\varphi_\delta(u_\epsilon(t)-\psi(t))+E\int_0^t\langle A(u_\epsilon,\cdot)-A(\psi,\cdot),F'_\delta(u_\epsilon-\psi)\rangle ds$$
$$-\frac{1}{\epsilon}E\int_0^t\langle[(u_\epsilon-\psi)^-]^{\tilde{q}-1},F'_\delta(u_\epsilon-\psi)\rangle ds\le E\int_0^t\langle -h^-,F'_\delta(u_\epsilon-\psi)\rangle ds.$$

Claim: a.e. $t\in[0,T]$ and P-a.s, $F'_\delta(u_\epsilon-\psi)$ converges to $-2(u_\epsilon-\psi)^-$ in V. Indeed, we have

$$F'_\delta(r)=\begin{cases}2r & if\ \ r\le-\delta,\\ -2\frac{r^3}{\delta^2}-4\frac{r^2}{\delta}\ \ if & -\delta\le r\le 0,\\ 0 & if\ \ r\ge 0.\end{cases}\tag{6}$$

Let us estimate $\|F'_\delta(u_\epsilon-\psi)+2(u_\epsilon-\psi)^-\|_V$,

$$\|F'_\delta(u_\epsilon-\psi)+2(u_\epsilon-\psi)^-\|_V$$
$$=\left(\int_D|F'_\delta(u_\epsilon(x)-\psi(x))+2(u_\epsilon(x)-\psi(x))^-|^2dx\right)^{\frac{1}{2}}$$
$$+\left(\int_D|\nabla F'_\delta(u_\epsilon(x)-\psi(x))+2\nabla(u_\epsilon(x)-\psi(x))^-|^pdx\right)^{\frac{1}{p}}.$$

We denote by B the set $\{-\delta\le u_\epsilon-\psi\le 0\}$. On one hand, one has

$$\int_D|F'_\delta(u_\epsilon(x)-\psi(x))+2(u_\epsilon(x)-\psi(x))^-|^2dx$$
$$=\int_B\left|-\frac{2}{\delta^2}(u_\epsilon(x)-\psi(x))^3-\frac{4}{\delta}(u_\epsilon(x)-\psi(x))^2-2(u_\epsilon(x)-\psi(x))\right|^2dx$$
$$\le C_2\int_D 8\delta^2dx=8C_2\delta^2 mes(D)\to 0 \text{ as } \delta\to 0.$$

On the other hand, setting $F=\{-\delta<u_\epsilon-\psi<0\}$ one has by the chain rule in the Sobolev spaces

$$\int_D|\nabla F'_\delta(u_\epsilon(x)-\psi(x))+2\nabla(u_\epsilon(x)-\psi(x))^-|^pdx$$
$$=\int_F\left|\frac{2}{\delta^2}\nabla(u_\epsilon(x)-\psi(x))^3+\frac{4}{\delta}\nabla(u_\epsilon(x)-\psi(x))^2+2\nabla(u_\epsilon(x)-\psi(x))\right|^pdx$$
$$\le\int_F\left|\left(\frac{6}{\delta^2}(u_\epsilon(x)-\psi(x))^2+\frac{8}{\delta}(u_\epsilon(x)-\psi(x))+2\right)\nabla(u_\epsilon(x)-\psi(x))\right|^pdx.$$

We have $|\frac{6}{\delta^2}(u_\epsilon(x)-\psi(x))^2+\frac{8}{\delta}(u_\epsilon(x)-\psi(x))+2|\mathbb{I}_F \leq 2\mathbb{I}_{\{-\delta<u_\epsilon-\psi<0\}} \to 0$ a.e. $x\in D$ as $\delta\to 0$ and

$$\left|\left(\frac{6}{\delta^2}(u_\epsilon(x)-\psi(x))^2+\frac{8}{\delta}(u_\epsilon(x)-\psi(x))+2\right)\mathbb{I}_F\nabla(u_\epsilon(x)-\psi(x))\right|^p \\ \leq 2|\nabla(u_\epsilon(x)-\psi(x))|^p.$$

This last function is in $L^1(D)$ and dominated convergence theorem ensures that $\int_D |\nabla F'_\delta(u_\epsilon(x)-\psi(x))+2\nabla(u_\epsilon(x)-\psi(x))^-|^p dx \to 0$.
Therefore a.e. $t\in[0,T]$ and P-a.s., one gets when $\delta\to 0$

- $\forall t\in[0,T],\quad \varphi_\delta(u_\epsilon(t)-\psi(t)) \longrightarrow \|(u_\epsilon-\psi)^-(t)\|^2_{L^2(D)}$,
- $$\langle A(u_\epsilon,\cdot)-A(\psi,\cdot),F'_\delta(u_\epsilon-\psi)\rangle \longrightarrow \langle A(u_\epsilon,\cdot)-A(\psi,\cdot),-2(u_\epsilon-\psi)^-\rangle \\ \geq -2\lambda_T\|(u_\epsilon-\psi)^-\|^2_H,$$

  since this last term is equal to

  $$2\langle A(\psi,\cdot)-A(u_\epsilon,\cdot),(\psi-u_\epsilon)^+\rangle \geq -2\lambda_T\|(\psi-u_\epsilon)^+\|^2_H$$

  thanks to $H_{2,2}$.
- $\langle -[(u_\epsilon-\psi)^-]^{\tilde{q}-1},F'_\delta(u_\epsilon-\psi)\rangle \to \langle -[(u_\epsilon-\psi)^-]^{\tilde{q}-1},-2(u_\epsilon-\psi)^-\rangle = 2\|(u_\epsilon-\psi)^-\|^{\tilde{q}}_{L^{\tilde{q}}(D)}$,
- $\langle -h^-,F'_\delta(u_\epsilon-\psi)\rangle \to \langle -h^-,2(u_\epsilon-\psi)\mathbb{I}_{\{u_\epsilon<\psi\}}\rangle = 2\langle h^-,(u_\epsilon-\psi)^-\rangle$.

Again, dominated convergence theorem ensures that for any t

$$E\|(u_\epsilon-\psi)^-(t)\|^2_{L^2(D)}+\frac{2}{\epsilon}E\int_0^t\|(u_\epsilon-\psi)^-(s)\|^{\tilde{q}}_{L^{\tilde{q}}(D)}ds \\ \leq 2E\int_0^t\langle h^-(s),(u_\epsilon-\psi)^-(s)\rangle ds+2\lambda_T\int_0^t\|(u_\epsilon-\psi)^-(s)\|^2_H ds. \tag{7}$$

To continue our proof, we will consider two cases.

- If $p\geq 2$ then $\tilde{q}=2$. By multiplying (7) by $\frac{1}{\epsilon}$, one gets

$$\frac{1}{2\epsilon}E\|(u_\epsilon-\psi)^-(T)\|^2_{L^2(D)}+\frac{1}{\epsilon^2}E\int_0^T\|(u_\epsilon-\psi)^-(s)\|^2_{L^2(D)}ds \\ \leq E\int_0^T\langle h^-(s),\frac{1}{\epsilon}(u_\epsilon-\psi)^-(s)\rangle ds+\frac{\epsilon\lambda_T^+}{\epsilon^2}\int_0^T\|(u_\epsilon-\psi)^-(s)\|^2_H ds.$$

Since $E\int_0^T \langle h^-(s), \frac{1}{\epsilon}(u_\epsilon - \psi)^-(s)\rangle ds \leq \frac{1}{2\epsilon^2}E\int_0^T \|(u_\epsilon - \psi)^-(s)\|^2_{L^2(D)}ds + \frac{1}{2}E\int_0^T \|h^-(s)\|^2_{L^2(D)}ds$, one has for $\epsilon \leq \frac{1}{4\lambda_T^+ + 1}$,

$$\frac{1}{2\epsilon}E\|(u_\epsilon - \psi)^-(T)\|^2_{L^2(D)} + \frac{1}{4\epsilon^2}E\int_0^T \|(u_\epsilon - \psi)^-(s)\|^2_{L^2(D)}ds$$
$$\leq E\int_0^T \|h^-(s)\|^2_{L^2(D)}ds.$$

Therefore $(\frac{1}{\epsilon}(u_\epsilon - \psi)^-)_{\epsilon>0}$ is bounded in $L^2(\Omega_T \times D)$.

- If $2 > p > 1$ then $\tilde{q} = p$. From Gronwall's lemma applied to (7), one gets

$$\begin{aligned}\frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^p_{L^p(\Omega_T\times D)} =&\frac{1}{\epsilon}E\int_0^T \|(u_\epsilon - \psi)^-(s)\|^p_{L^p(D)}ds\\ \leq& C(T)E\int_0^T |\langle h^-(s), (u_\epsilon - \psi)^-(s)\rangle| ds\\ \leq& C\|h^-\|_{L^{p'}(\Omega_T\times D)}\|(u_\epsilon - \psi)^-\|_{L^p(\Omega_T\times D)},\end{aligned}$$

hence $\frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^{p-1}_{L^p(\Omega_T\times D)} \leq \|h^-\|_{L^{p'}(\Omega_T\times D)}$. On the other hand, we have

$$\begin{aligned}\left\|\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1}\right\|_{L^{p'}(\Omega_T\times D)} =&\frac{1}{\epsilon}(E\int_0^T\int_D [(u_\epsilon - \psi)^-]^{(p-1)p'}dxds)^{\frac{1}{p'}}\\ =&\frac{1}{\epsilon}\|(u_\epsilon - \psi)^-\|^{p-1}_{L^p(\Omega_T\times D)}.\end{aligned}$$

Consequently, $(\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1})_{\epsilon>0}$ is bounded in $L^{p'}(\Omega_T \times D)$.

□

As a consequence the following lemma holds.

Lemma 4 *Under H_7, $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0,T], H))$.*

Proof Let $1 > \epsilon \geq \delta > 0$ and consider the process $u_\epsilon - u_\delta$, which satisfies the following equation

$$\begin{aligned}u_\epsilon(t) - u_\delta(t) + &\int_0^t (A(u_\epsilon,\cdot) - A(u_\delta,\cdot))\\ &+ \left(-\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} + \frac{1}{\delta}[(u_\delta - \psi)^-]^{\tilde{q}-1}\right)ds\\ =&\int_0^t (\widetilde{G}(u_\epsilon,\cdot) - \widetilde{G}(u_\delta,\cdot))dW(s).\end{aligned}$$

Applying Ito's formula with $F(t,v)=\frac{1}{2}\|v\|_H^2$, one gets for any $t\in[0,T]$

$$\begin{aligned}&\frac{1}{2}\|(u_\epsilon-u_\delta)(t)\|_H^2+\int_0^t\langle A(u_\epsilon,\cdot)-A(u_\delta,\cdot),u_\epsilon-u_\delta\rangle ds\\&\quad+\int_0^t\left\langle-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde q-1}+\frac{1}{\delta}[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-u_\delta\right\rangle ds\\&\quad=\int_0^t\langle(\widetilde G(u_\epsilon,\cdot)-\widetilde G(u_\delta,\cdot))dW(s),u_\epsilon-u_\delta\rangle+\frac{1}{2}\int_0^t|\widetilde G(u_\epsilon,\cdot)-\widetilde G(u_\delta,\cdot)|_Q^2ds.\end{aligned}$$

We argue as in the proof of (2) with $f_1=f_2$ and note that we need only to discuss the penalization term.

On one hand, using the monotonicity of the penalization operator, one deduces

$$\begin{aligned}&\int_0^t\left\langle-\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde q-1}+\frac{1}{\delta}[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-u_\delta\right\rangle ds\\&\quad\geq\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-u_\delta\rangle ds.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}&\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-u_\delta\rangle ds\\&\quad=\frac{\epsilon-\delta}{\epsilon\delta}\left(\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-\psi\rangle ds+\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},-(u_\delta-\psi)\rangle ds\right).\end{aligned}$$

Since $\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},-(u_\delta-\psi)\rangle ds\geq 0$, it holds that

$$\begin{aligned}\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-u_\delta\rangle ds&\geq\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},u_\epsilon-\psi\rangle ds\\&\geq-\frac{\epsilon-\delta}{\epsilon\delta}\int_0^t\langle[(u_\delta-\psi)^-]^{\tilde q-1},(u_\epsilon-\psi)^-\rangle ds.\end{aligned}$$

We distinguish two cases:

- If $p\geq 2$, then $\tilde q=2$. Since $(\frac{1}{\epsilon}(u_\epsilon-\psi)^-)_{\epsilon>0}$ is bounded in $L^2(\Omega_T\times D)$, we get

$$\begin{aligned}0\leq&\frac{\epsilon-\delta}{\epsilon\delta}E\int_0^T\langle[(u_\delta-\psi)^-]^{\tilde q-1},(u_\epsilon-\psi)^-\rangle ds\\&=(\epsilon-\delta)E\int_0^T\left\langle\frac{1}{\delta}(u_\delta-\psi)^-,\frac{1}{\epsilon}(u_\epsilon-\psi)^-\right\rangle ds\leq C\epsilon.\end{aligned}$$

- If $1 < p < 2$, then $\tilde{q} = p$. Since $(\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1})_{\epsilon>0}$ is bounded in $L^{p'}(\Omega_T \times D)$, we get

$$\begin{aligned}
0 \leq & \frac{\epsilon - \delta}{\epsilon\delta} E \int_0^T \langle [(u_\delta - \psi)^-]^{p-1}, (u_\epsilon - \psi)^- \rangle ds \\
= & \frac{\epsilon - \delta}{\epsilon} E \int_0^T \left\langle \frac{1}{\delta}[(u_\delta - \psi)^-]^{p-1}, (u_\epsilon - \psi)^- \right\rangle ds \\
\leq & \frac{\epsilon - \delta}{\epsilon} C \|(u_\epsilon - \psi)^-\|_{L^p(\Omega_T \times D)} \leq C\epsilon^{\frac{1}{p-1}}.
\end{aligned}$$

By arguments similar to the ones used to obtain (2), we deduce

$$E \sup_{t\in[0,T]} \|(u_\epsilon - u_\delta)(t)\|_H^2 \leq C(\epsilon + \epsilon^{\frac{1}{p-1}}) + C \int_0^T E \sup_{\tau\in[0,s]} \|(u_\epsilon - u_\delta)(\tau)\|_H^2 ds$$

and Gronwall's lemma ensures that $(u_\epsilon)_{\epsilon>0}$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0,T], H))$. □

3.1.3 At the limit as $\epsilon \to 0$

From Lemmas 2, 3 and 4, we deduce the following result.

Lemma 5 *There exist $u \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0,T], H)) \cap \mathcal{N}_W^2(0,T,H)$[2] and $(\rho, \chi) \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D)) \times L^{p'}(\Omega_T, V')$, each one predictable, such that the following convergences hold, up to sub-sequences denoted by the same way,*

$$u_\epsilon \rightharpoonup u \quad in \quad L^p(\Omega_T, V), \tag{8}$$

$$u_\epsilon \to u \quad in \quad L^2(\Omega, \mathcal{C}([0,T], H)), \tag{9}$$

$$A(u_\epsilon, \cdot) \rightharpoonup \chi \quad in \quad L^{p'}(\Omega_T, V'), \tag{10}$$

$$-\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} \rightharpoonup \rho, \quad \rho \leq 0 \quad in \quad L^{\tilde{q}'}(\Omega_T \times D). \tag{11}$$

Proof By compactness with respect to the weak topology in the spaces $L^p(\Omega_T, V)$, $L^{p'}(\Omega_T, V')$ and $L^{\tilde{q}'}(\Omega_T \times D)$, there exist $u \in L^p(\Omega_T, V)$, $\chi \in L^{p'}(\Omega_T, V')$ and $\rho \in L^{\tilde{q}'}(\Omega_T \times D)$ such that (8), (10) and (11) hold (for sub-sequences). Thanks to Lemma 4, we get the strong convergence of u_ϵ to u in $L^2(\Omega, \mathcal{C}([0,T], H)) \hookrightarrow L^2(\Omega_T \times D)$. Moreover,

- Since $u_\epsilon \in \mathcal{N}_W^2(0,T,H)$, a Hilbert space, $u \in \mathcal{N}_W^2(0,T,H)$ too.
- Since $(A(u_\epsilon, \cdot))_\epsilon$ is predictable with values in V' (cf. Remark 1), the same applies to χ.

[2] $\mathcal{N}_W^2(0,T,H)$ denotes the space of all predictable process of $L^2(\Omega_T, H)$ (see [17, p. 36]).

- Since $u_\epsilon, \psi \in \mathcal{N}^2_W(0,T,H)$, $-\dfrac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde{q}-1}$ is a predictable process with values in $L^{\tilde{q}'}(D)$. Hence ρ is a predictable process with values in $L^{\tilde{q}'}(D)$ and $\rho \le 0$ since the set of non positive functions of $L^{\tilde{q}'}(\Omega_T \times D)$ is a closed convex subset of $L^{\tilde{q}'}(\Omega_T \times D)$.

$\square$

Remark 8 *(initial condition and constraint).*

- Since u_ϵ converges to u in $L^2(\Omega, \mathcal{C}([0,T],H))$ then $u_\epsilon(0)=u_0$ converges to $u(0)$ in $L^2(\Omega,H)$ and $u(0)=u_0$ in $L^2(\Omega,H)$.
- Thanks to Lemma 3, we deduce that $(u_\epsilon-\psi)^- \to (u-\psi)^- = 0$ in $L^{\tilde{q}}(\Omega_T\times D)$ and $u \in K$.

Lemma 6 *Under* H_7, $\int_0^\cdot \widetilde{G}(u_\epsilon,\cdot)dW(s) \to \int_0^\cdot G(u,\cdot)dW(s)$ *in* $L^2(\Omega,\mathcal{C}([0,T],H))$ *when* $\epsilon \to 0$.

Proof By Burkhölder–Davis–Gundy's inequality [23, p.652], one gets

$$E \sup_{t\in[0,T]} |\int_0^t (\widetilde{G}(u_\epsilon,\cdot)-\widetilde{G}(u,\cdot))dW(s)|_H^2 \le 3E\int_0^T |\widetilde{G}(u_\epsilon,\cdot)-\widetilde{G}(u,\cdot)|_Q^2 ds$$
$$(\text{by using } H_3) \quad \le 3ME\int_0^T \|u_\epsilon-u\|_H^2 ds.$$

Since $u_\epsilon \to u$ in $L^2(\Omega,\mathcal{C}([0,T],H))$ with $u\in K$, one deduces

$$\int_0^\cdot \widetilde{G}(u_\epsilon,\cdot)dW(s) \to \int_0^\cdot \widetilde{G}(u,\cdot)dW(s) = \int_0^\cdot G(u,\cdot)dW(s) \quad in \quad L^2(\Omega,\mathcal{C}([0,T],H)).$$

$\square$

Lemma 7 *We have* $\rho(u-\psi)=0$ *a.e. in* Ω_T *and,* $\forall v \in K$, $\rho(u-v)\ge 0$ *a.e. in* Ω_T.

Proof On one hand, by Lemma 3, we have

$$0 \le -\frac{1}{\epsilon}E\int_0^t \langle [(u_\epsilon-\psi)^-]^{\tilde{q}-1}, u_\epsilon-\psi\rangle ds$$
$$= \frac{1}{\epsilon}E\int_0^t \|(u_\epsilon-\psi)^-(s)\|_{L^{\tilde{q}}}^{\tilde{q}} ds \le C\epsilon^{\tilde{q}'-1} \to 0.$$

On the other hand, by Lemma 5, we distinguish two cases:

- If $p\ge 2$ then $-\frac{1}{\epsilon}(u_\epsilon-\psi)^- \rightharpoonup \rho$ in $L^2(\Omega_T\times D)$ and $u_\epsilon-\psi \to u-\psi$ in $L^2(\Omega_T\times D)$ by Lemma 4. Hence $E\int_0^T\int_D \rho(u-\psi)dxdt = 0$ and $\rho(u-\psi)=0$ since the integrand is always non-positive.

- If $2 > p > 1$ then $-\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{p-1} \rightharpoonup \rho$ in $L^{p'}(\Omega_T \times D)$ and $u_\epsilon - \psi \to u - \psi$ in $L^p(\Omega_T \times D)$ by Lemma 4 and the same conclusion holds.

One finishes the proof by noticing that if $v \in K$, one has a.e. in Ω_T that,

$$\langle \rho, u - v \rangle = \overbrace{\langle \rho, u - \psi \rangle}^{=0} + \overbrace{\langle \rho, \psi - v \rangle}^{\geq 0} \geq 0.$$

□

Our aim now is to prove that $A(u, \cdot) = \chi$. We have for any $t \in [0, T]$

$$u_\epsilon(t) + \int_0^t (A(u_\epsilon, \cdot) - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - f)ds = u_0 + \int_0^t \widetilde{G}(u_\epsilon, \cdot)dW(s),$$

$$\text{and} \qquad u(t) + \int_0^t (\chi + \rho - f)ds = u_0 + \int_0^t \widetilde{G}(u, \cdot)dW(s).$$

Hence

$$u_\epsilon(t) - u(t) + \int_0^t \Big[(A(u_\epsilon, \cdot) - \chi) + \Big(-\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - \rho \Big) \Big] ds$$
$$= \int_0^t \widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(u, \cdot)dW(s).$$

Note that $(A(u_\epsilon, \cdot) - \chi) + (-\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - \rho) \in L^{p'}(\Omega_T, V')$ is predictable and that $\int_0^t \widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(u, \cdot)dW(s)$ is a square integrable $\mathcal{F}_t$−martingale. Thus, we can apply Ito's formula [20, Theorem 4.2 p. 65] to the process $u_\epsilon - u$ with $F(v) = \frac{1}{2}\|v\|_H^2$ to get

$$\frac{1}{2}\|(u_\epsilon - u)(t)\|_H^2 + \overbrace{\int_0^t \langle A(u_\epsilon, \cdot) - \chi, u_\epsilon - u \rangle ds}^{I_1}$$
$$+ \overbrace{\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1} - \rho, u_\epsilon - u \rangle ds}^{I_2}$$
$$= \overbrace{\int_0^t \langle \widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(u, \cdot)dW(s), u_\epsilon - u \rangle}^{I_3} + \overbrace{\frac{1}{2}\int_0^t |\widetilde{G}(u_\epsilon, \cdot) - \widetilde{G}(u, \cdot)|_Q^2 ds}^{I_4}.$$

Let us consider in the sequel a given $v \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0, T], H))$ and $t \in [0, T]$.

- Note that $I_1 = \int_0^t \langle A(u_\epsilon,\cdot), u_\epsilon\rangle ds - \int_0^t \langle A(u_\epsilon,\cdot), u\rangle ds - \int_0^t \langle \chi, u_\epsilon - u\rangle ds$ and

$$\begin{aligned}
\int_0^t \langle A(u_\epsilon,\cdot), u_\epsilon\rangle ds &= \int_0^t \langle A(u_\epsilon,\cdot) - A(v,\cdot), u_\epsilon - v\rangle ds \\
&\quad + \int_0^t \langle A(v,\cdot), u_\epsilon - v\rangle ds + \int_0^t \langle A(u_\epsilon,\cdot), v\rangle ds \\
(\lambda_T Id + A \text{ is T-monotone }) \quad &\geq \int_0^t \langle A(v,\cdot), u_\epsilon - v\rangle ds + \int_0^t \langle A(u_\epsilon,\cdot), v\rangle ds \\
&\quad - \lambda_T \int_0^t \|u_\epsilon - v\|_H^2 ds.
\end{aligned}$$

- $E(I_2) = E\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde{q}-1}, u_\epsilon - u\rangle ds - E\int_0^t \langle \rho, u_\epsilon - u\rangle ds$
  $\geq E\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde{q}-1}, \psi - u\rangle ds - E\int_0^t \langle \rho, u_\epsilon - u\rangle ds.$
- Since I_3 is a $\mathcal{F}_t$-martingale then $E(I_3) = 0$.
- Thanks to H_3 we have $E(I_4) \leq ME\int_0^t \|u_\epsilon(s) - u(s)\|_H^2 ds$.

By gathering the previous computation and taking the expectation, one has for any $t \in [0, T]$

$$\begin{aligned}
&\frac{1}{2}E\|(u_\epsilon - u)(t)\|_H^2 + E\int_0^t \Big[\langle A(v,\cdot), u_\epsilon - v\rangle + \langle A(u_\epsilon,\cdot), v-u\rangle - \langle \chi, u_\epsilon - u\rangle\Big]ds \\
&\quad + E\int_0^t \langle -\frac{1}{\epsilon}[(u_\epsilon-\psi)^-]^{\tilde{q}-1}, \psi - u\rangle ds - E\int_0^t \langle \rho, u_\epsilon - u\rangle ds \\
&\leq (M+\lambda_T)E\int_0^t \|u_\epsilon(s) - u(s)\|_H^2 ds.
\end{aligned}$$

By passing to the limit as $\epsilon \to 0$, thanks to Lemmas 5 and 7 and by setting $t = T$, we get

$$E\int_0^T \langle A(v,\cdot) - \chi, u - v\rangle ds \leq E\int_0^T \langle \rho, u - \psi\rangle ds = 0.$$

We are now in a position to use "Minty's trick" [25, Lemma 2.13 p.35] and deduce that $A(u,\cdot) = \chi$.
So, the conclusion of this section is: under assumption H_7, there exists a unique $(u,\rho) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$, both predictable, satisfying:

- $u \in L^2(\Omega, \mathcal{C}([0,T], H)) \cap K$ and $\rho \leq 0$.
- For any $t \in [0,T]$: $u(t) + \int_0^t (A(u,\cdot) + \rho - f)ds = u_0 + \int_0^t G(u,\cdot)dW(s)$.

- The first part of Lewy–Stampacchia's inequality holds:

$$\partial_t(u-\int_0^{\cdot} G(u,\cdot)dW)+A(u,\cdot)-f=-\rho\geq 0 \quad \text{in} \quad L^{\tilde{q}'}(\Omega_T\times D).$$

- $\langle \rho, u-\psi\rangle = 0$ a.e. in Ω_T and, for any $v\in K$, $\langle \rho, u-v\rangle \geq 0$ a.e. in Ω_T.

3.2 The second Lewy–Stampacchia's inequality with a regular h^-

The aim of this subsection is to prove the second part of Lewy–Stampacchia's inequality. For this, we used an idea inspired by [9]. Let u be the unique solution of Sect. 3.1 and denote by K_1 the closed convex set

$$K_1=\{v\in L^p(\Omega_T,V), \quad v\leq u \quad \text{a.e. in } D\times\Omega_T\}.$$

We recall that u satisfies

$$(f+h^-)-\partial_t(u-\int_0^{\cdot} G(u,\cdot)dW)-A(u,\cdot)=h^-+\rho, \quad \rho\leq 0, \quad \rho\in L^{\tilde{q}'}(\Omega_T\times D).$$

Consider the following auxiliary problem: $(z,\nu)\in L^p(\Omega_T,V)\times L^{\tilde{q}'}(\Omega_T,L^{\tilde{q}'}(D))$, predictable, such that

$$\begin{cases} i.)\ z\in L^2(\Omega,\mathcal{C}([0,T],H)), \quad z(0)=u_0 \quad \text{and} \quad z\in K_1,\\ ii.)\ \nu\geq 0,\ \langle \nu, z-u\rangle=0 \text{ and } \forall v\in K_1,\ \langle \nu, z-v\rangle\geq 0,\ \text{a.e. in } \Omega_T.\\ iii.)\ \text{P-a.s. and for any } t\in[0,T]:\\ z(t)+\int_0^t \nu ds+\int_0^t A(z,\cdot)ds=u_0+\int_0^t G(z,\cdot)dW(s)+\int_0^t (f+h^-)ds. \end{cases} \tag{12}$$

Note that the result of existence and uniqueness of the solution (z,ν) can be proved either by noting that $(-z,-\nu)$ is the solution to the above problem with data: $\tilde{f}=-f-h^-$, $\tilde{G}(v,\cdot)=-G(-v,\cdot)$, $\tilde{A}(v,\cdot)=-A(-v,\cdot)$, $\tilde{\psi}=-u$, $\tilde{h}^+=-\rho$ and $\tilde{h}^-=h^-$. This can also be obtained by cosmetic changes of what has been done in Sect. 3.1, by passing to the limit in the following penalized problem:

$$\begin{cases} z_\epsilon(t)+\int_0^t (A(z_\epsilon,\cdot)+\frac{1}{\epsilon}[(z_\epsilon-u)^+]^{\tilde{q}-1}-(f+h^-))ds=u_0+\int_0^t \widetilde{G}(z_\epsilon,\cdot)dW(s)\\ z_\epsilon(0)=u_0, \end{cases}$$

where $\widetilde{G}(z_\epsilon,\cdot)=G(\min(z_\epsilon,u),\cdot)$.
Moreover, $\partial_t(z-\int_0^{\cdot} G(z,\cdot)dW)+A(z,\cdot)-(f+h^-)=-\nu\leq 0$ in $L^{\tilde{q}'}(\Omega_T\times D)$ and z satisfies the following Lewy–Stampacchia's inequality:

$$\partial_t(z-\int_0^{\cdot} G(z,\cdot)dW)+A(z,\cdot)-f\leq h^- \quad \text{in} \quad L^{\tilde{q}'}(\Omega_T\times D).$$

We know already that $z \leq u$ and our aim is now to prove that $z = u$. For that, it is sufficient to prove that $z \geq \psi$. Indeed, let us assume for a moment that $z \geq \psi$, then

$$\langle \rho, u - z \rangle = \langle \rho, u - \psi \rangle + \langle \rho, \psi - z \rangle = \langle \rho, \psi - z \rangle \geq 0.$$

Thus we have $\langle \nu, z - u \rangle = 0$ and $\langle \rho - \nu, u - z \rangle = \langle \rho, u - z \rangle + \langle \nu, z - u \rangle \geq 0$. Therefore, applying Ito's energy to

$$\begin{aligned} u(t) - z(t) + \int_0^t \rho - \nu ds + \int_0^t A(u, \cdot) - A(z, \cdot) ds \\ = \int_0^t G(u, \cdot) - G(z, \cdot) dW(s) - \int_0^t h^- ds. \end{aligned}$$

yields for any $t \in [0, T]$

$$\begin{aligned} &\frac{1}{2}\|(u-z)(t)\|_H^2 + \int_0^t \underbrace{\langle A(u,\cdot) - A(z,\cdot), u - z\rangle}_{(\lambda_T Id + A \text{ is T-monotone}) \geq -\lambda_T \|u-z\|_H^2} ds + \underbrace{\int_0^t \langle \rho - \nu, u - z\rangle ds}_{\geq 0} \\ &\quad + \underbrace{\int_0^t \langle h^-, u - z\rangle ds}_{(u \geq z)\ \geq 0} \\ &= \int_0^t \langle (G(u,\cdot) - G(z,\cdot)) dW(s), u - z\rangle + \frac{1}{2}\int_0^t |G(u,\cdot) - G(z,\cdot)|_Q^2 ds. \end{aligned}$$

By similar arguments leading to (3), we conclude that $u = z$.
To conclude this subsection, we need to prove that $z \geq \psi$.
We know that $u \geq \psi$ so that $u - z = (u - z)^+ \geq (\psi - z)^+$ and $u \geq z + (\psi - z)^+ = z + (z - \psi)^-$.
Using $v = z + (z - \psi)^- \in K_1$ in (12)[ii.] yields $\langle \nu, (z - \psi)^- \rangle \leq 0$.

We have

$$\begin{aligned} z(t) - \psi(t) + \int_0^t \nu - h^+ ds + \int_0^t A(z, \cdot) - A(\psi, \cdot) ds \\ = u_0 - \psi(0) + \int_0^t G(z, \cdot) - G(\psi, \cdot) dW(s). \end{aligned}$$

As in the proof of Lemma 3, consider $\varphi_\delta(v) = \int_D F_\delta(v(x))dx$ and $S = \{z \le \psi\}$. Applying Ito's formula [20, Th. 5.3 p. 78] to the process $z - \psi$, one gets: $\forall t \in [0, T]$

$$\begin{aligned}
&\varphi_\delta(z(t) - \psi(t)) + \int_0^t \langle A(z,\cdot) - A(\psi,\cdot), F'_\delta(z-\psi)\rangle ds \\
&\quad + \int_0^t \langle \nu - h^+, F'_\delta(z-\psi)\rangle ds - \overbrace{\varphi_\delta(u_0 - \psi(0))}^{=0} \\
&= \int_0^t \langle G(z,\cdot) - G(\psi,\cdot)dW(s), F'_\delta(z-\psi)\rangle \\
&\quad + \frac{1}{2}\int_0^t |\sqrt{F''_\delta(z-\psi)}[G(z,\cdot) - G(\psi,\cdot)]|_Q^2 ds.
\end{aligned}$$

Note that

$$\begin{aligned}
\int_0^t |\sqrt{F''_\delta(z-\psi)}[G(z,\cdot) - G(\psi,\cdot)]|_Q^2 ds \le & \frac{8M}{3}\int_0^t \|z(s) - \psi(s)\|_H^2 \mathbb{I}_S ds \\
= & \frac{8M}{3}\int_0^t \|(z-\psi)^-(s)\|_H^2 ds.
\end{aligned}$$

Taking the expectation and passing to the limit when $\delta \to 0$,

- $\forall t \in [0, T], \quad E\varphi_\delta(z(t) - \psi(t)) \longrightarrow E\|(z-\psi)^-(t)\|^2_{L^2(D)}$,
- $E\int_0^t \langle A(z,\cdot) - A(\psi,\cdot), F'_\delta(z-\psi)\rangle ds \longrightarrow E\int_0^t \langle A(z,\cdot) - A(\psi,\cdot), -2(z-\psi)^-\rangle ds$
  $= 2E\int_0^t \langle A(\psi,\cdot) - A(z,\cdot), (\psi - z)^+\rangle ds$
  $\ge -2\lambda_T \int_0^t \|(\psi - z)^+\|_H^2 ds$,
- $E\int_0^t \langle \nu - h^+, F'_\delta(z-\psi)\rangle ds \to -2E\int_0^t \langle \nu - h^+, (z-\psi)^-\rangle ds$
  $= 2(E\int_0^t \langle h^+, (z-\psi)^-\rangle + \underbrace{\langle \nu, -(z-\psi)^-\rangle ds)}_{\text{(thanks (12)[ii.])}\ \ge 0} \ge 0.$

Those limits may be obtained by Lebesgue's theorem and, for any $t \in [0, T]$:

$$E\|(z-\psi)^-(t)\|_H^2 \le C\int_0^t E\|(z-\psi)^-(s)\|_H^2 ds.$$

Finally, Gronwall's lemma ensures that $\psi \le z$, and, as conclusion of this subsection, we get $z = u$. Hence, u satisfies the second part of Lewy–Stampacchia's inequality:

$$\partial_t\Big(u - \int_0^\cdot G(u,\cdot)dW\Big) + A(u,\cdot) - f \le h^- \quad \text{in} \quad L^{\tilde{q}'}(\Omega_T \times D).$$

From Sects. 3.1 and 3.2, we deduce the following theorem.

Theorem 2 *Under Assumptions* (H_1)*–*(H_6) *and assuming moreover that* $h^- \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ *is predictable, there exists a unique predictable stochastic process* $(u, k) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ *such that:*

i. $u \in L^2(\Omega, \mathcal{C}([0, T], H)) \cap K$, $u(0) = u_0$.
ii. $k \le 0$ *and* $\forall v \in K$, $\langle k, u - v\rangle \ge 0$ *a.e. in* Ω_T.
iii. P-a.s, for all $t \in [0, T]$,

$$u(t) + \int_0^t kds + \int_0^t A(u, \cdot)ds = u_0 + \int_0^t G(u, \cdot)dW(s) + \int_0^t fds.$$

iv. The following Lewy–Stampacchia's inequality holds:

$$0 \le \partial_t(u - \int_0^\cdot G(u, \cdot)dW) + A(u, \cdot) - f \le h^- = \left(f - \partial_t(\psi - \int_0^\cdot G(\psi, \cdot)dW) - A(\psi, \cdot)\right)^- .$$

3.3 Proof of the main theorem in the general case

First, we prove the following lemma which allows us to pass from the regular to the general case.

3.3.1 Density result in the positive cone of the dual

Lemma 8 *The positive cone of* $L^p(\Omega_T, V) \cap L^2(\Omega_T, L^2(D))$ *is dense in the positive cone of* $L^{p'}(\Omega_T, V')$*. Moreover, the positive cone of predictable elements of* $L^p(\Omega_T, V) \cap L^2(\Omega_T, L^2(D))$ *is dense in the positive cone of predictable elements of* $L^{p'}(\Omega_T, V')$.

By a truncation argument, the same result holds for the positive cone of $L^p(\Omega_T, V) \cap L^{p'}(\Omega_T, L^{p'}(D))$ (resp. predictable).

Proof Since the proof of the lemma is mainly based on monotone arguments, it is similar to the one proposed in [13, Lemma 4.1] where one has just to add the predictable assumption to the spaces of type $L^r(0, T, X)$ in [13, Lemma 4.1] if needed. □

3.3.2 Proof of Theorem 1

Let $h^- \in (L^{p'}(\Omega_T, V'))^+$ predictable. Thanks to Lemma 8, there exists $h_n \in L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ predictable and non negative such that

$$h_n \longrightarrow h^- \quad \text{in} \quad L^{p'}(\Omega_T, V').$$

Associated with h_n, denote the following f_n by,

$$f_n = \partial_t(\psi - \int_0^\cdot G(\psi, \cdot)dW) + A(\psi, \cdot) + h^+ \\ -h_n, \ h^+ \in (L^{p'}(\Omega_T, V'))^+ \text{ predictable too.}$$

Note that $f_n \in L^{p'}(\Omega_T, V')$ is predictable and f_n converges strongly to f in $L^{p'}(\Omega_T, V')$.
Denote by (u_n, k_n) the sequence of solutions given by Theorem 2 where h^- is replaced by h_n.
By Lewy–Stampacchia's inequality, one has $0 \leq -k_n \leq h_n$.
For any $\varphi \in L^p(\Omega_T, V)$, it holds that

$$\begin{aligned} E\int_0^T |\langle k_n, \varphi\rangle| ds &\leq E\int_0^T \langle -k_n, \varphi^+\rangle ds + E\int_0^T \langle -k_n, \varphi^-\rangle ds \\ &\leq E\int_0^T \langle h_n, \varphi^+\rangle ds + E\int_0^T \langle h_n, \varphi^-\rangle ds \\ &\leq 2\|h_n\|_{L^{p'}(\Omega_T, V')}\|\varphi\|_{L^p(\Omega_T, V)}. \end{aligned}$$

Since $(h_n)_n$ converges to h in $L^{p'}(\Omega_T, V')$, one gets that $(h_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$ and therefore $(k_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$.
Let $n \in \mathbb{N}^*$ and applying Ito's energy formula to the process u_n, one gets for any $t \in [0, T]$

$$\begin{aligned} \frac{1}{2}\|u_n(t)\|_H^2 + \int_0^t \langle A(u_n, \cdot), u_n\rangle ds =& \frac{1}{2}\|u_0\|_H^2 + \int_0^t \langle -k_n, u_n\rangle ds + \int_0^t \langle f_n, u_n\rangle ds \\ &+ \int_0^t \langle G(u_n, \cdot)dW(s), u_n\rangle + \frac{1}{2}\int_0^t |G(u_n, \cdot)|_Q^2 ds. \end{aligned}$$

Since f_n converges to f in $L^{p'}(\Omega_T, V')$, it holds that $(f_n)_n$ is bounded independently of n in $L^{p'}(\Omega_T, V')$. Therefore, by Young's inequality, we get

$$E\int_0^T |\langle f_n - k_n, u_n\rangle| ds \leq \frac{\alpha}{2} E\int_0^T \|u_n(s)\|_V^p ds + C\|f_n - k_n\|_{L^{p'}(\Omega_T, V')}^{p'}.$$

By Burkhölder–Davis–Gundy's inequality and Young's inequality, there exists $\delta > 0$ such that

$$\begin{aligned} E\left[\sup_{t\in[0,T]} \left|\int_0^t \langle G(u_n, \cdot)dW(s), u_n\rangle\right|\right] \leq& \frac{3\delta}{2} E \sup_{t\in[0,T]} \|u_n(t)\|_H^2 \\ &+ \frac{3M}{2\delta} E\int_0^T \|u_n(s)\|_H^2 ds + \frac{3}{2\delta}\|l\|_{L^1(\Omega_T)}. \end{aligned}$$

With a convenient choice of δ (*e.g.* $\delta = \frac{1}{4}$) and using $H_{2,1}$, $H_{3,2}$, one deduces

$$E \sup_{t\in[0,T]} \|u_n(t)\|_H^2 + E\int_0^T \|u_n(s)\|_V^p ds \leq C(1 + E\int_0^T \sup_{\tau\in[0,s]} \|u_n(\tau)\|_H^2 ds).$$

By using Gronwall's lemma, one concludes that $(u_n)_n$ is bounded in $L^p(\Omega_T, V) \cap L^2(\Omega, L^\infty(0, T, H))$.
Now, we present the following lemma about the strong convergence of $(u_n)_n$.

Lemma 9 *$(u_n)_n$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H))$.*

Proof Let $m, n \in \mathbb{N}^*$ and $\epsilon > 0$. For any $t \in [0, T]$ and P-a.s, we have

$$u_n(t) - u_m(t) + \int_0^t (k_n - k_m)ds + \int_0^t (A(u_n, \cdot) - A(u_m, \cdot))ds$$
$$= \int_0^t (G(u_n, \cdot) - G(u_m, \cdot))dW(s) + \int_0^t (f_n - f_m)ds.$$

Applying Ito's energy formula, one gets for any $t \in [0, T]$,

$$\frac{1}{2}\|(u_n - u_m)(t)\|_H^2 + \int_0^t \langle A(u_n, \cdot) - A(u_m, \cdot), u_n - u_m\rangle ds$$
$$= - \int_0^t \langle k_n - k_m, u_n - u_m\rangle ds + \int_0^t \langle f_n - f_m, u_n - u_m\rangle ds$$
$$+ \int_0^t \langle (G(u_n, \cdot) - G(u_m, \cdot))dW(s), u_n - u_m\rangle + \frac{1}{2}\int_0^t |G(u_n, \cdot) - G(u_m, \cdot)|_Q^2 ds.$$

Similarly to the proof of Lemma 1, one deduces

$$E \sup_{t\in[0,T]} \|(u_n - u_m)(t)\|_H^2 \le C\|f_n - f_m\|_{L^{p'}(\Omega_T,V')}\|u_n - u_m\|_{L^p(\Omega_T,V)}.$$

Since f_n converges strongly to f in $L^{p'}(\Omega_T, V')$ and $(u_n)_n$ is bounded in $L^p(\Omega_T, V)$, it holds that

$$E \int_0^T \langle f_n - f_m, u_n - u_m\rangle ds \le \|f_n - f_m\|_{L^{p'}(\Omega_T,V')}\|u_n - u_m\|_{L^p(\Omega_T,V)} \le C\epsilon,$$

for big values of n and m. Therefore $(u_n)_n$ is a Cauchy sequence in the space $L^2(\Omega, \mathcal{C}([0, T], H)$. □

Since $(u_n)_n$ is bounded sequence in $L^p(\Omega_T, V)$ of predictable processes, Remark 1 and $\text{H}_{2,3}$ yield that $(A(u_n, \cdot))_n$ is a bounded sequence in $L^{p'}(\Omega_T, V')$ of predictable processes.
By compactness with respect to the weak topology in the spaces $L^p(\Omega_T, V)$ and $L^{p'}(\Omega_T, V')$, there exist $u \in L^p(\Omega_T, V)$, $\chi \in L^{p'}(\Omega_T, V')$ and $k \in L^{p'}(\Omega_T, V')$, each one being predictable, such that (up to sub-sequences denoted by the same way)

$$u_n \rightharpoonup u \quad \text{in} \quad L^p(\Omega_T, V), \tag{13}$$
$$A(u_n, \cdot) \rightharpoonup \chi \quad \text{in} \quad L^{p'}(\Omega_T, V'), \tag{14}$$
$$k_n \rightharpoonup k \quad \text{in} \quad L^{p'}(\Omega_T, V'). \tag{15}$$

Thanks to Lemma 9, we have the strong convergence of u_n to u in $L^2(\Omega, \mathcal{C}([0,T],H))$ thus in $L^2(\Omega_T, L^2(D))$ and $u \in \mathcal{N}_W^2(0,T,H)$.
Since $(-k_n) \in (L^{p'}(\Omega_T, V'))^+$ and $-k_n \rightharpoonup k$ in $L^{p'}(\Omega_T, V')$, we deduce that $-k \in (L^{p'}(\Omega_T, V'))^+$. Indeed, let $\varphi \in L^p(\Omega_T, V)$, $\varphi \geq 0$ then

$$E\int_0^T \langle -k, \varphi\rangle ds = \lim_{n\to\infty} E\int_0^T \langle -k_n, \varphi\rangle ds \geq 0.$$

Remark 9 (Initial condition and constraint).

- Since u_n converges to u in $L^2(\Omega, \mathcal{C}([0,T],H))$ with $u_n(0) = u_0$, one has that $u(0) = u_0$.
- Since K is a closed convex subset of $L^p(\Omega_T, V)$, it holds that $u \in K$.

Similarly to the proof of Lemma 6, one gets

$$\int_0^{\cdot} G(u_n,\cdot)dW(s) \to \int_0^{\cdot} G(u,\cdot)dW(s) \ in \ L^2(\Omega, \mathcal{C}([0,T],H)) \quad when \quad n\to\infty.$$

So, at the limit, we have a.s. and for any $t \in [0,T]$

$$u(t) + \int_0^t k ds + \int_0^t \chi ds = u_0 + \int_0^t G(u,\cdot)dW(s) + \int_0^t f ds.$$

For any $n \in \mathbb{N}^*$, we have a.s. and for any $t \in [0,T]$

$$u_n(t) + \int_0^t k_n ds + \int_0^t A(u_n,\cdot)ds = u_0 + \int_0^t G(u_n,\cdot)dW(s) + \int_0^t f_n ds.$$

Note that $(A(u_n,\cdot)-\chi) + (k_n - k) + (f_n - f) \in L^{p'}(\Omega_T, V')$ is predictable and $\int_0^t (G(u_n,\cdot) - G(u,\cdot))dW(s)$ is a square integrable $\mathcal{F}_t-$martingale. We can apply Ito's formula [20, Theorem 4.2 p. 65] to the process $u_n - u$ with $F(v) = \frac{1}{2}\|v\|_H^2$ to get

$$\frac{1}{2}\|(u_n-u)(t)\|_H^2 + \int_0^t \langle A(u_n,\cdot)-\chi, u_n-u\rangle ds + \int_0^t \langle k_n - k, u_n - u\rangle ds$$
$$= \overbrace{\int_0^t \langle (G(u_n,\cdot)-G(u,\cdot))dW(s), u_n-u\rangle}^{I_1(t)} + \overbrace{\frac{1}{2}\int_0^t |G(u_n,\cdot)-G(u,\cdot)|_Q^2 ds}^{I_2(t)}$$
$$+ \overbrace{\int_0^t \langle f_n - f, u_n - u\rangle ds}^{I_3(t)}.$$

Thanks to Lemma 7, $\langle k_n, u_n - \psi\rangle = 0$ and one has

$$\langle k_n - k, u_n - u\rangle = \langle k_n, u_n - u\rangle - \langle k, u_n - u\rangle$$
$$= \langle k_n, u_n - \psi\rangle + \langle k_n, \psi - u\rangle - \langle k, u_n - u\rangle = \langle k_n, \psi - u\rangle - \langle k, u_n - u\rangle.$$

Therefore

$$E\int_0^T \langle k_n - k, u_n - u\rangle ds = E\int_0^T \langle k_n, \psi - u\rangle ds - \overbrace{E\int_0^T \langle k, u_n - u\rangle ds}^{\to 0}$$
$$\longrightarrow E\int_0^T \langle k, \psi - u\rangle ds \geq 0.$$

Since f_n converges strongly to f in $L^{p'}(\Omega_T, V')$, (13) ensures that $E(I_3(T)) \to 0$. Similarly to the last part of Sect. 3.1, one has: $E(I_1(t)) = 0$, $E(I_2(T)) \to 0$ and, for any $v \in L^p(\Omega_T, V) \cap L^2(\Omega, \mathcal{C}([0,T], H))$,

$$E\int_0^T \langle k, \psi - u\rangle ds + E\int_0^T \langle A(v, \cdot) - \chi, u - v\rangle ds \leq 0. \tag{16}$$

By setting $v = u$ in (16), one has $E\int_0^T \langle k, \psi - u\rangle ds \leq 0$.
Therefore $E\int_0^T \langle k, \psi - u\rangle ds = 0$.
Since $-k \in (L^{p'}(\Omega_T, V'))^+$, $-k(t,\omega) \in (V')^+$ a.e. in Ω_T. ‘
Hence, $\langle k(s,\omega), \psi - u\rangle \geq 0$ and $\langle k(s,\omega), \psi - u\rangle = 0$ a.e. in Ω_T.
By (16), we get $E\int_0^T \langle A(v,\cdot) - \chi, u - v\rangle ds \leq 0$, then, using “Minty trick” one concludes that $\chi = A(u, \cdot)$.

Let $v \in K$, then a.e. Ω_T, we have $\langle k, u - v\rangle = \langle k, u - \psi\rangle + \langle k, \psi - v\rangle \geq 0$. We deduce the existence result of Theorem 1 for general f. At last, Lewy–Stampacchia's inequality is a consequence of the passage to the limit in the one satisfied by u_n.

4 Examples of numerical illustrations

Consider the following problem:

$$\begin{cases} du - \alpha u_{xx} ds + k ds = f ds + \sigma u dW & in \quad]0,1[\times\Omega\times]0,1], \\ u(t=0) = u_0 \geq 0 & \text{in } L^2(0,1),\ a.s., \\ u \geq 0 & in \quad [0,1]\times\Omega\times[0,1], \\ u(0,t) = u(1,t) = 0 & on \quad \Omega\times[0,1], \\ \langle k, u\rangle = 0 \ \text{ and } \ k \leq 0 & in \quad \Omega\times[0,1] \end{cases} \tag{17}$$

where $\alpha > 0$, $\sigma \in \mathbb{R}$ and f is a smooth function. By [23, Thm 5] or the above Theorem 1, there exists a unique solution (u, k) to Problem (17) in the sense of Definition 2 with $p = 2$ and $D =]0, 1[$. Moreover, The following Lewy–Stampacchia's inequality holds

$$0 \leq \partial_t\Big(u - \sigma \int_0^{\cdot} u dW\Big) - \alpha u_{xx} - f \leq h^- = f^-.$$

Note that thanks to Remark 3, this basic situation of a constraint of positivity with a vanishing stochastic reaction term at 0 can be an illustration of a more general situation.

In this section, we propose some numerical illustrations of the solution of the obstacle problem (17) and, at the same time, we compare them to the numerical solution of the free problem *i.e* the stochastic heat equation when the constraint $u \geq 0$ is ignored.

To the best of the author's knowledge, there doesn't exist in the literature numerical studies of stochastic obstacle problems. Inspired by previous sections, our aim is to present some numerical illustrations of the stochastic obstacle problem (17) *via* a penalty method, i.e. an approximation by the family $(P_\epsilon)_{\epsilon>0}$ of penalized problems:

$$P_\epsilon : \begin{cases} u^\epsilon(t) - \int_0^t (\alpha u^\epsilon_{xx} + \frac{1}{\epsilon}[(u^\epsilon)^-] + f) ds = u_0 + \sigma \int_0^t u^\epsilon(s) dW(s) \\ u^\epsilon(0) = u_0, \\ u^\epsilon(0,t) = u^\epsilon(1,t) = 0 \quad on \quad \Omega\times[0,1]. \end{cases} \tag{18}$$

For that, one needs a suitable choice of the small parameter ϵ compatible with the space and time discretization steps.

Let us denote by $\Delta t = \dfrac{1}{N}$ the time step of the uniform discretization of the time-interval $[0, 1]$, $\{t_0, \cdots, t_N\}$ are the points of this discretization. Similarly, $\Delta x = \dfrac{1}{M}$ is the uniform space step discretization of the space-interval $[0, 1]$ and $\{x_0, \cdots, x_M\}$ are the points of the space discretization. Then, following what is usually done in the deterministic case (see e.g. [26]), one sets $\epsilon = \Delta t = (\Delta x)^2$ to ensure the convergence of the scheme (19) below to the solution of (17).

Denote by U^i_j the approximate solutions at time t_i, computed at x_j when U_0 is given by the initial condition, *via* $U_0 = \{u_0(x_1), \cdots, u_0(x_{M-1})\}$.

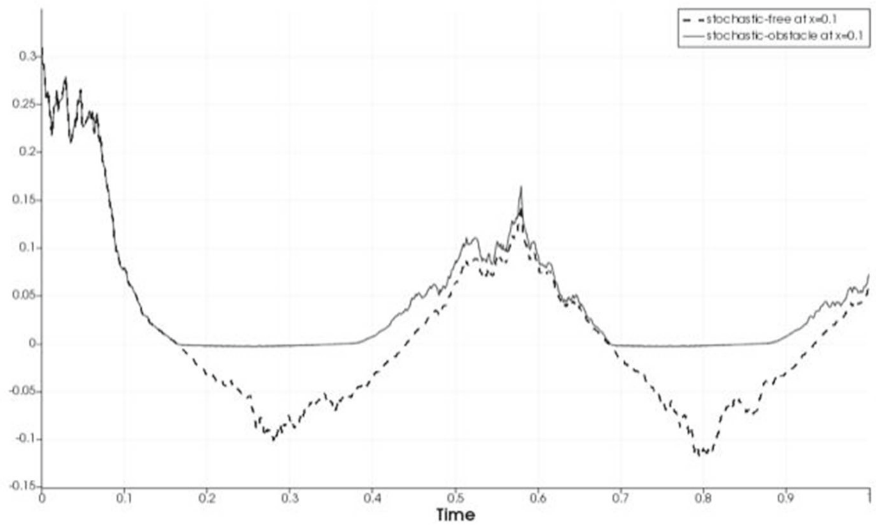

Fig. 1 Pathwise trajectory at x = 0.1

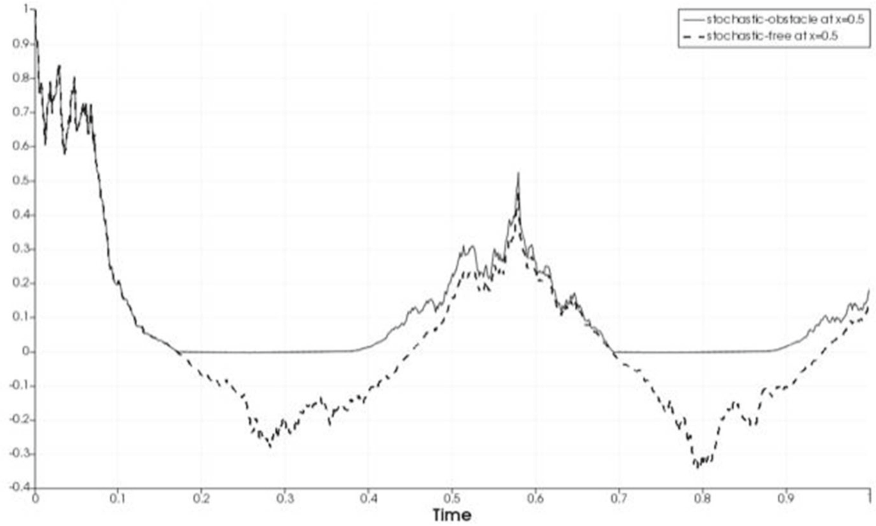

Fig. 2 Pathwise trajectory at x = 0.5

We consider the following approximate discretized problem obtained *via* a penalty method, a stochastic "Saul'yev scheme" (see [12]) *i.e*

$$\begin{cases} U_i^j = \dfrac{1}{1+\beta}(\beta U_{i-1}^{j+1} + (1-\beta)U_{i-1}^j + \beta U_i^{j-1}) + \dfrac{\sigma U_{i-1}^j}{1+\beta}(W(t_i) - W(t_{i-1}) \\ \qquad + \dfrac{\Delta t}{1+\beta} f(t_{i-1}, x_j) + \dfrac{\Delta t}{\epsilon(1+\beta)}(U_{i-1}^j)^-, \quad 1 \le j \le M-1, \quad 1 \le i \le N, \\ U_0^j = u_0(x_j), \quad 1 \le j \le M-1 \\ U_i^0 = U_i^{M+1} = 0, \quad 0 \le i \le N, \end{cases} \tag{19}$$

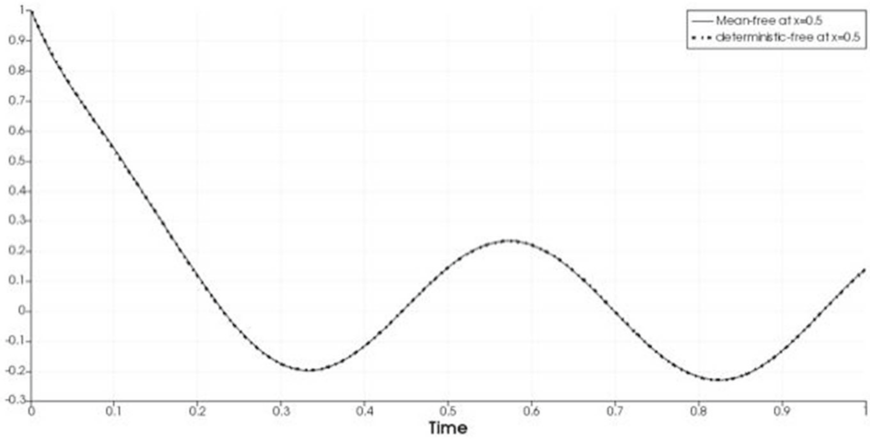

Fig. 3 Mean of 5000 samples paths and deterministic solution at x = 0.5 for the heat equation

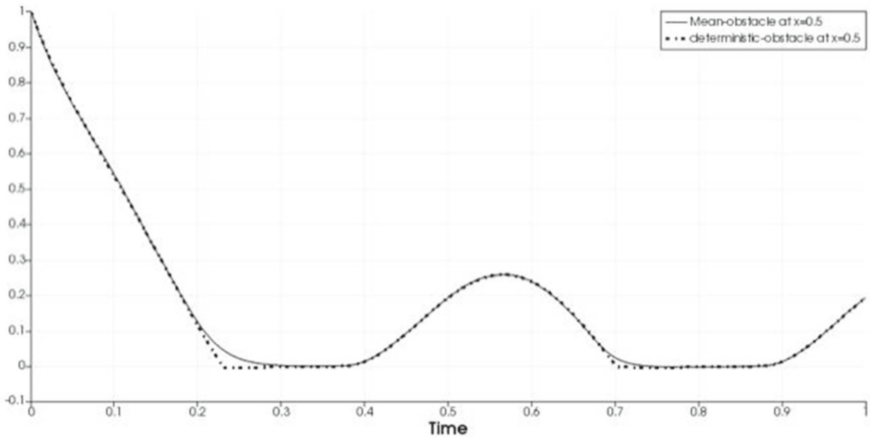

Fig. 4 Mean of 5000 samples paths and deterministic solution at x = 0.5 with the constraint

where $\beta = \alpha \dfrac{\Delta t}{(\Delta x)^2}$.

The numerical simulations of Figs. 1, 2, 3, and 4 are implemented with the free software *Scilab* and the following data: $u_0(x) = \sin(\pi x), \alpha = 1, f(x, t, \omega) = 3\cos(4\pi t), \sigma = 2, N = 900$ and $M = 30$.

- In the first two figures, we present pathwise trajectories of the penalized problem (19) in full line and of the free stochastic problem (without the penalization-term) in dotted-line. Fig. 1 represents the simulation at point $x = 0.1$, close to the boundary, and Fig. 2 at point $x = 0.5$ in the middle of the domain.
  One can see that, as expected, the trajectories of the free and obstacle problems are the same before the first time-contact with the obstacle. When the constraint is active for Problem (19), the solution is equal to the constraint 0, else it is positive.

- In the last two figures, we present the simulation of the deterministic problem (*i.e.* when $\sigma = 0$) in dotted-line and the mean of 5000 trajectories of the stochastic problem in full line. Figure 3 represents the simulations of the problem without constraint and Fig. 4 is concerned by Problem (19).
  As expected for the linear heat equation (case of Fig. 3), the mean of the stochastic paths coincides with the solution to the deterministic problem. The situation is slithy different for the problem with constraint. Indeed, even if the constraint is deterministic, the penalization, and the Lagrange multiplier at the limit, induces a non linear term. Thus, the mean and the deterministic solution may differ.

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5 Appendix

5.1 Itô's formula with non-nul trace on the boundary

We are interested in this subsection in replacing the assumption $\psi \in L^p(\Omega_T, V)$ of H4 by $\psi \in L^p\big(\Omega_T, W^{1,p}(D) \cap L^2(D)\big) \cap L^{\max(p,p')}\big(\Omega, C([0,T], L^2(D))\big)$ with a non-positive trace on the boundary.[3] This situation appears for example if A is a Leray–Lions type differential operator of the form $A(u,t,\omega) = -div(a(t,\omega,x,\nabla u)) + b(t,\omega,x,u)$ that can be defined on $\Omega_T \times W^{1,p}(D) \cap L^2(D)$ with values in V' by: $(t,\omega) \in \Omega_T \quad a.e.$,

$$\langle A(u,t,\omega), v\rangle = \int_D a(t,\omega,x,\nabla u)\nabla v dx + \int_D b(t,\omega,x,u) v dx, \quad \forall v \in V.$$

In order to be able to follow the same steps of our demonstration, only two major points need to be adapted: the first one is in the proof of Lemma 2 where choosing $v^* = \psi$ is not possible anymore; the second one is in the proof of Lemma 3 since, in this new situation, $u - \psi$ is not with values in V anymore and the classical Itô formula no longer applies. The other modifications are minor ones based on embeddings of V into some Lebesgue's spaces that still hold when replacing $W_0^{1,p}(D)$ by $W^{1,p}(D)$. Concerning the question of v^*, one can chose for it the solution to the problem

$$\partial_t\Big[v^* - \int_0^\cdot \tilde G(v^*,\cdot)dW\Big] - \Delta_p v^* = \partial_t\Big[\psi - \int_0^\cdot \tilde G(\psi,\cdot)dW\Big] - \Delta_p \psi = \bar f$$

associated with Dirichlet boundary conditions, $v^*(0) = \psi(0)$ and where, by assumption $\bar f$ is a predictable process in $L^{p'}(\Omega_T, V')$. v^* exists with the convenient regularity and one still need to prove that $v^* \geq \psi$ to have it in K and use it in the proof. This is

[3] Note that the pathwise continuity assumption can be implicit thanks to arguments similar to [13, Lemma 4.7].

achieved by applying formally Itô's formula to the process $(v^* - \psi)^-$ where

$$d(v^* - \psi) - [\Delta_p v^* - \Delta_p \psi]dt = [\tilde{G}(v^*, \cdot) - \tilde{G}(\psi, \cdot)]dW.$$

The question related to Lemma 3 is similar since the proof is based on the possibility to apply Itô's formula to the process $(u_\epsilon - \psi)^-$ where

$$\begin{aligned} d(u_\epsilon - \psi) + [A(u_\epsilon, \cdot) - A(\psi, \cdot)]dt - \frac{1}{\epsilon}[(u_\epsilon - \psi)^-]^{\tilde{q}-1}dt \\ = hdt + [\tilde{G}(u_\epsilon, \cdot) - \tilde{G}(\psi, \cdot)]dW. \end{aligned} \tag{20}$$

In both situation, one has a predictable process X, being $v^* - \psi$ in the first case and $u_\epsilon - \psi$ is the second one, with values in $W^{1,p}(D) \cap L^2(D)$ and not *a priori* V, such that $dX + Adt = GdW$ where A is with values in V' and G in $L_Q(L^2(D))$. This is not a classical situation and Itô's formula associated with the negative-part should apply since X has a positive trace on the boundary of D and thus X^- is with values in V.

For any positive integer n, denote by Φ_n the function $x \mapsto \min(1, nd(x, \partial D))$. This is a sequence of bounded 1-Lipschitz continuous functions that converges a.e. to 1 in D. Thus, for any $u \in W^{1,p}(D) \cap L^2(D)$, the product $u\Phi_n$ is in V and if moreover u belongs to V, then $u\Phi_n$ converges to u in V.

Indeed, convergences of $u\Phi_n$ to u in $L^p(D) \cap L^2(D)$ and $\Phi_n \nabla u$ to ∇u in $L^p(D)$ are just applications of Lebesgue Theorem, and

$$\begin{aligned} u\nabla\Phi_n &= nu\nabla d(\cdot, \partial D)1_{\{0<d(\cdot,\partial D)<\frac{1}{n}\}}, \\ |u\nabla\Phi_n| &\leq n|u|1_{\{0<d(\cdot,\partial D)<\frac{1}{n}\}} \leq \frac{|u|}{d(\cdot, \partial D)}1_{\{0<d(\cdot,\partial D)<\frac{1}{n}\}}, \end{aligned}$$

and $u\nabla\Phi_n$ tends to 0 in $L^p(D)$ since, by Hardy's inequality, $\frac{u}{d(\cdot,\partial D)}$ is in $L^p(D)$. Since the product $\Phi_n A$ for A in V' is $\varphi \in V \mapsto \langle A, \Phi_n \varphi \rangle$, one gets that

$$dX\Phi_n + \Phi_n Adt = \Phi_n GdW$$

with now $X\Phi_n$ with values in V so that Itô's formula is applicable, in particular with the function F_δ introduced in (5). Thus, with the notations of the proof of Lemma 3

$$\begin{aligned} &\varphi_\delta(X\Phi_n) + \int_0^t \langle A, \Phi_n F'_\delta(X\Phi_n)\rangle ds \\ &= \int_0^t \Phi_n G F'_\delta(X\Phi_n)dW + \frac{1}{2}\int_0^t Tr(F''_\delta(X\Phi_n)\{\Phi_n G\}Q\{\Phi_n G\}^*)ds. \end{aligned}$$

Note that $F'_\delta(X\Phi_n) = F'_\delta(-X^-\Phi_n)$ and since X^- is in V, passing to the limit in n is possible. Thus, the desired Itô formula is proved for X and Theorem 1 holds when one assumes that the obstacle may have a non-positive value on the boundary of D.

5.2 On bilateral problems

We are interested in this subsection in saying few words about the situation of double obstacles problems. First, let us precise assumptions on obstacles.

H_4^* : ψ_1, ψ_2 satisfy H_4 with $\psi_2 \geq \psi_1$ a.e. in : $D \times \Omega_T$.
H_5^* : Assumption H_5 is satisfied by both obstacles ψ_i $i = 1, 2$:

$$h_i = h_i^+ - h_i^- = f - \partial_t\Big(\psi_i - \int_0^{\cdot} G(\psi_i, \cdot)dW\Big) - A(\psi_i, \cdot)$$

with the associated regularity information.
H_6^* : u_0 satisfies the constraints, *i.e.* $\psi_2(0) \geq u_0 \geq \psi_1(0)$.
H_7^* : h_1^-, h_2^+ are predictable non negative elements of $L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$.

The convex set of admissible functions becomes

$$K_{\psi_1}^{\psi_2} = \{v \in L^p(\Omega_T, V), \quad \psi_1(x,t,\omega) \leq v(x,t,\omega) \leq \psi_2(x,t,\omega) \quad a.e. \, in \quad D \times \Omega_T\},$$

and note that $K_{\psi_1}^{\psi_2}$ is not empty since $\psi_i \in K_{\psi_1}^{\psi_2}, \quad i = 1, 2$.
The idea is to follow the same strategy than the one used in the one obstacle case. In other words, we consider the same assumptions on the operator A, the multiplicative noise G and update the other assumptions. The corresponding penalized problem is

$$\begin{cases} u_\epsilon(t) + \int_0^t (A(u_\epsilon, \cdot) - \frac{1}{\epsilon}[(u_\epsilon - \psi_1)^-]^{\tilde{q}-1} + \frac{1}{\epsilon}[(u_\epsilon - \psi_2)^+]^{\tilde{q}-1} - f)ds \\ \qquad\qquad = u_0 + \int_0^t \widetilde{G}(u_\epsilon, \cdot)dW(s) \\ u_\epsilon(0) = u_0, \end{cases} \tag{21}$$

where $\widetilde{G}(u_\epsilon, \cdot) = G(\max(\min(u_\epsilon, \psi_2), \psi_1), \cdot)$, which satisfies properties similar to G and behaves formally as an additive stochastic source on the free-set where the constraints are violated.
By cosmetic changes of what has been done in Sect. 3 and by noticing that the penalized term is the sum of two parts with disjoint supports, one can prove the boundedness of the two parts of penalized terms independently. Then, passing to the limit in (21) to prove the existence of a solution. Finally, we can prove the two parts of Lewy–Stampacchia inequalities independently by adapting the arguments used in Sect. 3.2; and the one of the proof of Lemma 1 to get the uniqueness result. Thus, one gets

Theorem 3 *Under Assumptions (H_1)–(H_3) and (H_i^*, i=4,5,6,7), there exists a unique predictable stochastic process* $(u, \rho_1, \rho_2) \in L^p(\Omega_T, V) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D)) \times L^{\tilde{q}'}(\Omega_T, L^{\tilde{q}'}(D))$ *such that:*

i. $u \in L^2(\Omega, \mathcal{C}([0,T], H)) \cap K_{\psi_1}^{\psi_2}$, $u(0) = u_0$.
ii. $-\rho_1, \rho_2 \geq 0$ *and* $\forall v \in K_{\psi_1}^{\psi_2}$, $\langle \rho_i, u - v\rangle \geq 0, \quad i = 1, 2$ *a.e. in* Ω_T.

iii. P-a.s, for all $t \in [0, T]$,

$$u(t) + \int_0^t (\rho_1 + \rho_2) ds + \int_0^t A(u, \cdot) ds = u_0 + \int_0^t G(u, \cdot) dW(s) + \int_0^t f ds.$$

iv. The following Lewy–Stampacchia's inequality holds:

$$\begin{aligned} -h_2^+ &= -\left(f - \partial_t(\psi_2 - \int_0^\cdot G(\psi_2, \cdot) dW) - A(\psi_2, \cdot)\right)^+ \\ &\le \partial_t(u - \int_0^\cdot G(u, \cdot) dW) + A(u, \cdot) - f \\ &\le h_1^- = \left(f - \partial_t(\psi_1 - \int_0^\cdot G(\psi_1, \cdot) dW) - A(\psi_1, \cdot)\right)^- . \end{aligned}$$

The reader interested in relaxing Assumption H_7^* could be inspired by the strategy of [18], for example.

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